Solution to Problem #9

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Notice that $F_{n+1}(x) = x^{F_n(x)} = \mathrm{e}^{F_n(x)\ln x}$. The Chain Rule thus yields

$$\begin{aligned} F'_{n+1}(x) &= \mathrm{e}^{F_n(x)\ln x}(F'_n(x)\ln x + (F_n(x)/x)) \\ &= F_{n+1}(x)(F'_n(x)\ln x + (F_n(x)/x)). \qquad (1) \end{aligned}$$

Consequently, using the fact that $F_1(x) = x$ and $F'_1(x) = 1$,

$$F'_2(x) = F_2(x)(F'_1(x)\ln x + (F_1(x)/x)) = x^x(\ln x + 1).$$

Similarly,

$$F'_3(x) = F_3(x)(F'_2(x)\ln x + (F_2(x)/x)) = x^{(x^x)}x^x((\ln x + 1)\ln x + (1/x)).$$

Finally,

$$F'_{n+1}(1) = F_{n+1}(1)F_n(1) = 1.$$

Notes:

- *The last step could be realized without using the induction principle, due to the cancellation of $F'_n(1)\ln 1$ in the formula (1).*