Solution to Problem #8

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By the assumption,

$$\begin{aligned} x^3 - x^2 + x - 2 &= (x-\alpha)(x-\beta)(x-\gamma) \\ &= x^3 - (\alpha+\beta+\gamma)x^2 + (\alpha\beta+\alpha\gamma+\beta\gamma)x - \alpha\beta\gamma. \end{aligned}$$

By equating the coefficients,

$$\alpha+\beta+\gamma = 1, \tag{1}$$
$$\alpha\beta+\alpha\gamma+\beta\gamma = 1, \tag{2}$$
$$\alpha\beta\gamma = 2. \tag{3}$$

Squaring the equation (1), the equation (2) then yields

$$1 = (\alpha+\beta+\gamma)^2 = \alpha^2+\beta^2+\gamma^2+2(\alpha\beta+\alpha\gamma+\beta\gamma) = \alpha^2+\beta^2+\gamma^2+2,$$

and hence $\alpha^2+\beta^2+\gamma^2 = -1$. In particular, α, β, and γ cannot be all real.

Notes:

- *The coefficients of* $(x-\alpha)(x-\beta)(x-\gamma)$ *are examples of* symmetric polynomials *in* α, β, γ*: to interchange these variables does not affect them. The linear combinations of these polynomials generate* all *symmetric polynomials in* α, β, γ*, in particular,* $\alpha^2+\beta^2+\gamma^2$.
- *The result may be generalized to polynomials of any degree. In particular, to evaluate a symmetric polynomial at the roots of given polynomial of the same degree may be done without factorizing this last.*