Solution to Problem #7

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We want to evaluate $I = \displaystyle\int_0^{\pi/2} \frac{\sin^n x}{\sin^n x + \cos^n x}\,\mathrm{d}x$. Let $u = \dfrac{\pi}{2} - x$. Notice that $\sin u = \cos x$, while $\cos u = \sin x$. The change of variables thus yields $I = -\displaystyle\int_{\pi/2}^{0} \frac{\cos^n u}{\cos^n u + \sin^n u}\,\mathrm{d}u$, that is, after renaming the variable of integration,

$$I = \int_0^{\pi/2} \frac{\cos^n x}{\sin^n x + \cos^n x}\,\mathrm{d}x.$$

Therefore, $2I = \displaystyle\int_0^{\pi/2} 1\,\mathrm{d}x = \pi/2$, and hence $I = \pi/4$ for any value of n.

Notes:

- *The argument works indeed for any $n \in \mathbb{R}$. Of course, the value $I = \pi/4$ is easy to guess by taking $n = 0$.*
- *The problem is a special case of the following result: under suitable conditions on f, $\displaystyle\int_0^a \frac{f(x)}{f(x) + f(a-x)}\,\mathrm{d}x = \frac{a}{2}$.*