Solution to Problem #5

March 5, 2012

(a) Let $P(x) = ax^2 + bx + c$, where $a \neq 0$. Then,

$$P(x) = a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right) = a\left(\left(x + \frac{b}{2a}\right)^2 + \frac{c}{a} - \frac{b^2}{4a^2}\right),$$

so $P(x) = 0$ if and only if $x + \dfrac{b}{2a} = \pm\dfrac{\sqrt{b^2 - 4ac}}{2a}$, where the square-root may be complex.

(b) Let $Q(x) = a_3x^3 + a_2x^2 + a_1x + a_0$, where $a_3 \neq 0$. Since $Q(x)$ goes to ∞ when $x \to \infty$, while it goes to $-\infty$ when $x \to -\infty$, by continuity, there exists an $x_0 \in \mathbb{R}$ such that $Q(x_0) = 0$.

Notes:

- *We based our solution in (b) on the Intermediate Value Theorem, whose proof requires completeness of the real line. An alternative is to observe that if a complex number z_0 satisfies $Q(z_0) = 0$, then $Q(\bar{z}_0) = 0$, the coefficients of Q being real. In particular, the degree of a polynomial on $\mathbb{R}$ without real root must be even.*

  However in the above argument, we have used the Fundamental Theorem of Algebra for ensuring that at least one complex root exists. Its proof is even harder than studying completeness of the line. For avoiding this problem, since we are dealing with a polynomial of degree 3, we could instead deduce the existence of a complex root from Cardano's formula, whose proof is elementary, but tricky.

- *Congratulations to Shaikhah Abdulla Tahmoun Al-Yammahi, who submitted a perfect solution using the Intermediate Value Theorem. A honorable mention is also given to Wadha Khalifa Al Falasi, who submitted an almost perfect solution using conjugates, with only a sign mistake in (a).*