Solution to Problem #4

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We will proceed by induction on n. For $n = 0$, $F_m F_1 - F_0 F_m = (-1)^0 F_m$ is clearly satisfied, since $F_0 = 0$ and $F_1 = 1$. Assume that

$$F_m F_{n+1} - F_n F_{m+1} = (-1)^n F_{m-n} \text{ for all } m,$$

where n is arbitrarily fixed. We want to show that

$$F_m F_{n+2} - F_{n+1} F_{m+1} \stackrel{?}{=} (-1)^{n+1} F_{m-(n+1)} \text{ for all } m.$$

Let us denote by L the left-hand side of this last, desired relation. The relation $F_{n+2} = F_{n+1} + F_n$ yields $L = F_m F_{n+1} + F_m F_n - F_{n+1} F_{m+1}$. Then, the inductive hypothesis on $F_m F_{n+1}$ implies

$$L = [(-1)^n F_{m-n} + F_n F_{m+1}] + F_m F_n - F_{n+1} F_{m+1}.$$

Since $F_m + F_{m+1} = F_{m+2}$, it follows that

$$L = -(F_{m+1} F_{n+1} - F_n F_{m+2}) + (-1)^n F_{m-n}.$$

The inductive hypothesis (with $m+1$ in the role of m) assures that $F_{m+1} F_{n+1} - F_n F_{m+2} = (-1)^n F_{(m+1)-n}$, and hence

$$L = (-1)^{n+1} (F_{m+1-n} - F_{m-n}).$$

Since $F_{m+1-n} = F_{m-n} + F_{m-n-1}$, it follows that $L = (-1)^{n+1} F_{m-(n+1)}$, as desired.

Note: The proven result is known as d'Ocagne's identity. *However, I have not checked in the literature if a simpler proof is presented.*