Solution to Problem #3

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Observe that the series is a telescoping series. Indeed, if S_n is the nth partial sum, then

$$S_n = \sum_{k=0}^{n-1} \left[\int_0^{k+1} e^{-x^2}\, dx - \int_0^{k} e^{-x^2}\, dx \right] = \int_0^n e^{-x^2}\, dx.$$

Therefore, the series converges if and only if $\displaystyle\int_0^\infty e^{-x^2}\, dx$ converges. But,

$$\int_0^\infty e^{-x^2}\, dx = \int_0^1 e^{-x^2}\, dx + \int_1^\infty e^{-x^2}\, dx$$

and $0 \leqslant e^{-x^2} \leqslant e^{-x}$ if $x \geqslant 1$. Since $\displaystyle\int_1^\infty e^{-x}\, dx$ converges, by the Comparison Test $\displaystyle\int_1^\infty e^{-x^2}\, dx$ converges and so does $\displaystyle\int_0^\infty e^{-x^2}\, dx$. Now, Let $I = \displaystyle\int_0^\infty e^{-x^2}\, dx$. Then,

$$I^2 = \left(\int_0^\infty e^{-x^2}\, dx \right) \left(\int_0^\infty e^{-y^2}\, dy \right) = \int_0^\infty \int_0^\infty e^{-\left(x^2+y^2\right)}\, dxdy.$$

Further, setting $x = r\cos(\theta)$ and $y = r\sin(\theta)$, I^2 simplifies to

$$I^2 = \int_0^{\pi/2} \int_0^\infty re^{-r^2}\, drd\theta.$$

The inner integral converges to $1/2$. Hence, $I^2 = \pi/4$ and so, $I = \sqrt{\pi}/2$.

Note: As observed by Dr. Philippe, the convergence of $\int_0^\infty e^{x^2} dx$ follows directly from the computation of I^2 (whose integrand is positive) and Fubini's theorem. Though this last argument saves computations, it is not elementary.