Solution to Problem #1

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(a) Let $f(x) = a^x - x^a$. Notice that $f(a) = 0$. Hence, for $f(x)$ not to change sign, a must be a critical number: since $f'(x) = a^x \ln a - ax^{a-1}$, $f'(a)$ exists and must satisfy $f'(a) = 0$. It yields $a^a \ln a - aa^{a-1} = 0$, that is, $\ln a = 1$, which is equivalent to $a = \mathrm{e}$.

For the sake of completeness, let us prove that $y = \mathrm{e}^x$ and $y = x^{\mathrm{e}}$ do not cross. Letting $f(x) = \mathrm{e}^x - x^{\mathrm{e}}$, notice that $f''(\mathrm{e}) = \mathrm{e}^{\mathrm{e}-1} > 0$, so there is a local minimum at e. Notice also that $f(0) = 1$. If c is a critical number, then $\mathrm{e}^c - \mathrm{e}c^{\mathrm{e}-1} = 0$, and hence $f(c) = c^{\mathrm{e}}(\frac{\mathrm{e}}{c} - 1)$. In particular, there is no positive local maximum for $x > \mathrm{e}$, while there is no negative local minimum for $x < \mathrm{e}$. It forces $f(x)$ not to cross the real axis.

(b) If x^x reaches its minimum at a, then $a^a \leqslant x^x$ for all $x \geqslant 0$. In other words, $(1/x)^{1/a} \leqslant (1/a)^{1/x}$. By what precedes (with $1/x$ in the role of x and $1/a$ in the role of a), we conclude $1/a = \mathrm{e}$, as desired.

Note:

It is instructive to mention the following, elegant solution proposed by Prof. Raghib. If $y = a^x$ does not cross $y = x^a$, then either $a^x \leqslant x^a$, or $a^x \geqslant x^a$. In other words, either $\ln a/a \leqslant \ln x/x$, or $\ln a/a \geqslant \ln x/x$. In any case, $\ln x/x$ must reach an extremum at a. But $\ln x/x$ has one maximum only, at e; it does not have a minimum. Therefore, $a = \mathrm{e}$,—and $y = \mathrm{e}^x$ does not cross $y = x^{\mathrm{e}}$.