Problem #6

March 4, 2012

Difficulty: medium
Prerequisite: vectors in $\mathbb{R}^n$ (Calculus III or Linear Algebra)

(a) Let $u = (a_1, \ldots, a_n)$ and $v = (b_1, \ldots, b_n)$ be vectors in $\mathbb{R}^n$. The *norm* of u is defined as

$$\|u\| = \sqrt{a_1^2 + \cdots + a_n^2}\,,$$

while the *scalar product* of u and v is defined as

$$u \cdot v = a_1 b_1 + \cdots + a_n b_n.$$

It is easy to express the norm in terms of the scalar product only: $\|u\| = \sqrt{u \cdot u}$. We now ask the converse: express the scalar product $u \cdot v$ using norms only, where v is not necessarily equal to u.

(b) Prove the following generalization of Pythagorean Theorem:

$$\|u\|^2 + \|v\|^2 = \frac{1}{2}(\|u + v\|^2 + \|u - v\|^2).$$