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Hyperbolic Geometry (Math 491), Final Exam

2009 Fall

Name____________________I.D.___________Grade______(out of 40)

1. (5 marks) Mark the following statements True or False with a short proof or a counter example.

   (a) Each Euclidean point in $\boldsymbol{H}$ is a hyperbolic point.

   (b) A line perpendicular to the x-axis is a hyperbolic line.

   (c) The upper half plane $\boldsymbol{H}$ is an open disc in $\overline{\mathbb{C}}$.

   (d) The group of Möbius transformations $M\ddot{o}b(\boldsymbol{H})$ acts transitively on the set of triples of distinct hyperbolic points in $\boldsymbol{H}$.

   (e) The group of general Möbius transformations $M\ddot{o}b(\overline{\mathbb{C}})$ is the same as the group $Homeo^c(\overline{\mathbb{C}})$ of all homeomorphisms taking circles to circles in $\mathbb{C}$.

   (a) (3 marks) Using the cross ratio, determine whether the following four points lie on a circle in the Riemann sphere $\overline{\mathbb{C}}$:

$$z_1 = 0, z_2 = 3 + 2i, z_3 = 3 - 2i, z_4 = 1 + i.$$

   (b) (3 marks) Find a Möbius transformation $m(z) \in M\ddot{o}b(\boldsymbol{H})$ taking the hyperbolic line l (the two ending points at infinity are $x = 3$ and $y = -9$) to the hyperbolic line I (the positive imaginary axis).

2. (6 marks) Consider the stereographic projection

$$\phi : \boldsymbol{S}^2 - \{(0,0,1)\} \to \mathbb{R}^2$$

Show that

$$\phi(x_1, x_2, x_3) = (\frac{x_1}{1-x_3}, \frac{x_2}{1-x_3}) \text{ for each } (x_1, x_2, x_3) \in \boldsymbol{S}^2 - \{(0,0,1)\}.$$

3. (5 marks) Prove that the group $M\ddot{o}b(\boldsymbol{H})$ of all hyperbolic isometries of the hyperbolic plane $\boldsymbol{H}$ acts transitively on the set S if and only if there is an element $s_0 \in S$, for any element $s \in S$, there exists a Möbius transformation $m \in M\ddot{o}b(\boldsymbol{H})$ such that $m(s) = s_0$.

(a) (3 marks) Find a Möbius transformation $m(z) \in M\ddot{o}b(\overline{\mathbb{C}})$ taking the unit circle S^1 to the circle $\overline{\mathbb{R}}$.

(b) (3 marks) Determine whether the Möbius transformation $m(z)$ in (a) takes the unit disc D to the hyperbolic plane $\boldsymbol{H}$. If no, modify the Möbius transformation $m(z)$ in (a) so that it takes the unit disc D to the hyperbolic plane $\boldsymbol{H}$?

4. (6 marks) The group $M\ddot{o}b(\boldsymbol{H})$ of all hyperbolic isometries of the hyperbolic plane $\boldsymbol{H}$ is generated by $f(z) = az + b$ (with $a > 0, b \in \mathbb{R}$), $K(z) = \frac{-1}{z}$ and $B(z) = -\overline{z}$.

   (a) (5 marks) Determine whether or not the following Möbius transformation lies in $M\ddot{o}b(\boldsymbol{H})$. If yes, express $m(z)$ in terms of $f(z)$, $K(z)$ and $B(z)$.

$$m(z) = \frac{6z+1}{2-3z}.$$

   (b) (1 mark) Give the geometric interpretation of the general Möbius transformation

$$B(z) = -\overline{z}.$$

5.

(a) (3 marks) Find the hyperbolic distance $d_{\boldsymbol{H}}(P,Q)$ between two points $P = 2i$ and $Q = 8i$.Is this hyperbolic distance $d_{\boldsymbol{H}}(P,Q)$ the same as the Euclidean distance $d_{\boldsymbol{E}}(P,Q)$ between two points P and Q?

(b) (3 marks) Determine whether or not the following hyperbolic distances are the same as the hyperbolic distance $d_{\boldsymbol{H}}(P,Q)$.

(i) $d_{\boldsymbol{H}}(A,B) : A = -1 + 2i, B = -1 + 8i$.

(ii) $d_{\boldsymbol{H}}(C,D) : C = 2 + i, D = 8 + i$.

(iii) $d_{\boldsymbol{H}}(E,F) : E = 5 + 4i, F = 5 + 10i$.