United Arab Emirates University
Department of Mathematical Sciences

Numerical Analysis I **Midterm Exam** **November 3, 2009**
Name: **Student no:** **Serial no:**

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1. **(7 points) Perform** $(\frac{11}{3} - \frac{13}{11}) - \frac{16}{33}$, **using 3 digits chopping arithmetic. Compute the actual error.**

$11/3 = 3.66 \quad 13/11 = 1.18 \quad 16/33 = 0.484$

$3.66 - 1.18 = 2.48$

$2.48 - 0.484 = 1.99$

Exact $\left(\frac{11}{3} - \frac{13}{11}\right) - \frac{16}{33} = \frac{121-39}{33} - \frac{16}{33} = \frac{66}{33} = 2$

Error $= 2 - 1.99 = 0.01$

2. **(5 points) Find the largest interval in which** $p*$ **must lie to approximate** $\sqrt{2}$ **with relative error at most** 10^{-4}.

$\left|\frac{p-p^*}{p}\right| < 10^{-4} \Rightarrow \left|\frac{\sqrt{2}-p^*}{\sqrt{2}}\right| < 10^{-4}$

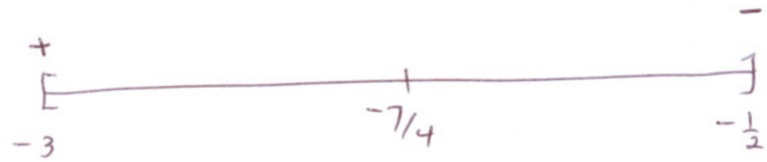
$\Rightarrow |p^*-p| < \sqrt{2}\,10^{-4}$

$-\sqrt{2}\,10^{-4} < p^*-p < \sqrt{2}\,10^{-4}$

$\sqrt{2} - \sqrt{2}\,10^{-4} < p^* < \sqrt{2} + \sqrt{2}\,10^{-4}$

3. (7 points) Find the roots of $f(x) = x(x+2)(x+1)^2(x-1)^3(x-2)$. **To which zero of** $f(x)$ **does the bisection method converge when applied on the interval** $[-3, \frac{-1}{2}]$. **SHOW YOUR WORK**

Zeros of $f(x)$ are $0, -2, -1, 1, 2$



$0, 1, 2 \notin [-3, -\frac{1}{2}]$

$f(-3) = -3(-1)(-2)^2(-4)^3(-5)$

$\quad\quad - \quad - \quad + \quad - \quad - \quad > 0$

$f(-\frac{1}{2}) = - \; + \; + \; - \; - \; < 0$

$P_1 = \dfrac{-3 + -\frac{1}{2}}{2} = -\dfrac{7}{4}$

$f(-7/4) = - \; + \; + \; - \; - \; < 0$

$\Rightarrow p \in [-3, -7/4] \qquad -1 \notin [-3, -7/4]$

$\Rightarrow$ converges to -2

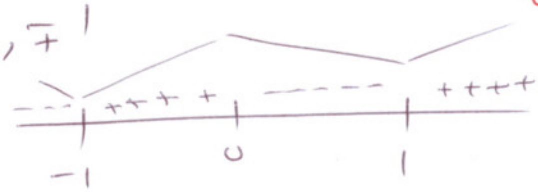
4. (8 points) Show that $g(x) = \frac{1}{16}x^4 - \frac{1}{8}x^2 + \frac{1}{2}$ has a unique fixed point on $[0, 1]$.

g cts. on $[0,1]$ ✓

$g'(x) = \frac{1}{4}x^3 - \frac{1}{4}x = \frac{1}{4}x(x^2-1) = 0$ ✓

$\Rightarrow x = 0, \mp 1$

g'



g decreasing on $[0,1]$ ✓

$\max = g(0) = \frac{1}{2}$

$\min = g(1) = \frac{1}{16} - \frac{1}{8} + \frac{1}{2} = -\frac{1}{16} + \frac{1}{2} = \frac{7}{16}$ ✓✓

$[\min g, \max g] \subseteq [0,1] \Rightarrow g(x) \subseteq [0,1]$

For the uniqueness

$|g'| = \left|\frac{1}{4}x(x^2-1)\right| = \frac{1}{4}x(1-x^2)$ ✓✓✓ on $[0,1]$

$|x| \leq 1 \qquad |1-x^2| \leq 1$

$|g'| \leq \frac{1}{4} \cdot 1 \cdot 1 = \frac{1}{4}$

$\Rightarrow$ the fixed point is unique

5. (6 points) Use the fixed point iteration to approximate the solution of $x^2 - x + \frac{1}{4} = 0$. Start with $p_0 = \frac{1}{4}$ and do 3 iteration.

$g(x) = x = x^2 + \frac{1}{4}$

$P_1 = \left(\frac{1}{4}\right)^2 + \frac{1}{4} = \frac{1}{64} + \frac{1}{4} = \frac{17}{64} = 0.2656$

$P_2 = \qquad = 0.3205$

$P_3 = \qquad = 0.3527$

6. (8 points) Show that $P_n = \frac{1}{n^2}$ converges linearly. How large must n such that $|p_n - p| < 5 \times 10^{-2}$.

$P_n \longrightarrow 0$

$\frac{|P_{n+1} - p|}{|P_n - p|^\alpha} = \frac{\left|\frac{1}{(n+1)^2} - 0\right|}{\left|\frac{1}{n^2} - 0\right|^\alpha} = \frac{(n^2)^\alpha}{(n+1)^2} \longrightarrow 1$

provided $\alpha = 1$.

$|P_n - p| < 5 + 10^{-2}$

$\left|\frac{1}{n^2} - 0\right| < \frac{5}{10^2}$

$n^2 > \frac{10^2}{5} = 20$

$\Rightarrow n = 5$

7. **(9 points) Use the nodes $x_0 = 0, x_1 = \frac{1}{2}$ and $x_2 = 1$ to construct Lagrange polynomial of degree 2 for $f(x) = e^{-2x}$. Estimate the error on** $[0, 1]$.

$$L_0(x) = \frac{(x-\frac{1}{2})(x-1)}{(0-\frac{1}{2})(0-1)} = 2(x-\tfrac{1}{2})(x-1)$$

$$L_1(x) = \frac{(x-0)(x-1)}{(\frac{1}{2}-0)(\frac{1}{2}-1)} = -4x(x-1)$$

$$L_2(x) = \frac{(x-0)(x-\frac{1}{2})}{(1-0)(1-\frac{1}{2})} = 2x(x-\tfrac{1}{2})$$

$$P_3(x) = 2(x-\tfrac{1}{2})(x-1)\cdot 1 - 4x(x-1)e^{-1} + 2x(x-\tfrac{1}{2})e^{-2}$$

$$f(x) - P_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)(x-x_1)(x-x_2) = \frac{f^{(3)}(\xi)}{3!}x(x-\tfrac{1}{2})(x-1)$$

$$f(x) = e^{-2x} \qquad f' = -2e^{-2x} \qquad f'' = 4e^{-2x} \qquad f''' = -8e^{-2x}$$

$$\left|\frac{f^{(3)}(\xi)}{3!}\right| = \frac{|-8e^{-2\xi}|}{6} \le \frac{8}{6}\cdot 1 = 4/3$$

$|x| \le 1$ $\qquad 0 \le x \le 1 \Rightarrow -\frac{1}{2} \le x - \frac{1}{2} \le \frac{1}{2} \Rightarrow |x-\frac{1}{2}| \le \frac{1}{2}$

$\qquad 0 \le x \le 1 \Rightarrow -1 \le x-1 \le 0 \Rightarrow |x-1| \le 1$

$$|f(x) - P_3(x)| \le \frac{4}{3}\cdot 1\cdot\frac{1}{2}\cdot 1 = 2/3$$