United Arab Emirates University
Department of Mathematical Sciences
Math 310 (Real Analysis)
Spring 2010, Midterm Exam
April 14^{th}, 2010

Name: Solution Keys Student Number:

[Q.1, 3 pts] Use the definition to prove that $f(x) = \cos x$ is everywhere continuous function.
Hint: $\cos x - \cos y = -2\sin(\frac{1}{2}(x+y))\sin(\frac{1}{2}(x-y)),\ \forall x, y \in \mathbb{R}$.

Let $c \in \mathbb{R}$, and $\varepsilon > 0$. Want $\delta > 0$ s.t.
$\forall x$; $|x-c| < \delta \implies |\cos x - \cos c| < \epsilon$.

But $|\cos x - \cos c| = 2\,|\sin(\frac{1}{2}(x+c))|\,|\sin\frac{1}{2}(x-c)|$

$\leq 2\cdot(1)(\frac{1}{2})\,|x-c|$; since $|\sin x| \leq |x|$.

$= |x-c| < \delta < \varepsilon$.

$\therefore$ choose $\boxed{\delta = \epsilon}$ (or $\delta \leq \epsilon$)

[Q.2, 3 pts] Use sequential criterion to prove that $\lim_{x\to 0}\sin(\frac{1}{x})$ does not exist.

Let $x_n = \frac{1}{n\pi} \longrightarrow 0$, but $\sin(\frac{1}{x_n}) = \sin(n\pi) = 0$; $\forall n$.

and let $y_n = \frac{1}{\frac{\pi}{2} + 2n\pi} \longrightarrow 0$

but $\sin(\frac{1}{y_n}) = \sin(\frac{\pi}{2} + 2n\pi) = 1$; $\forall n$.

4 pts

[Q.3, 4+2+2 pts](a) Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Show that there exists $x^* \in [a, b]$, such that $f(x^*) = \sup\{f(x),\ x \in [a, b]\}$ i.e. x^* is an absolute maximum.

Let $M = \sup\{f(x) ; x \in [a,b]\}$ exists.

$\forall n,\ \exists x_n \in [a,b]$ s.t.

$$M - \frac{1}{n} < f(x_n) < M \quad ; \text{ since } M \text{ is the sup.}$$

$\{x_n\}_{n=1}^{\infty}$ is bounded $\Rightarrow \exists$ subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ Convergent

$$x_{n_k} \to l \in [a,b] \quad \therefore f(x_{n_k}) \longrightarrow f(l).$$

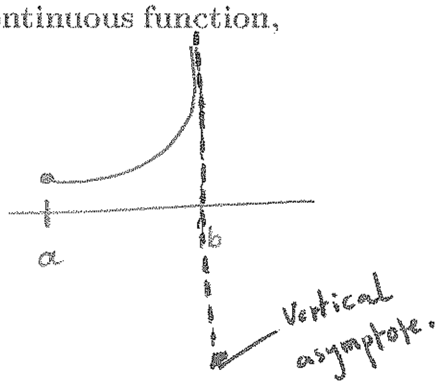
$\therefore \forall k \quad M - \frac{1}{n_k} < f(x_{n_k}) < M$

$\therefore$ By squeezing $f(x_{n_k}) \to M = l$.

2 pts

(b) Give an example to show if $f : [a, b) \to \mathbb{R}$ is a continuous function, then f may not have an absolute maximum.

Consider the example



more precisely $f(x) = \dfrac{1}{2-x}$; $x \in [0,2)$

has no abs. max.

(c) Let $f : [-2, 2] \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} x \sin \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

2pts

Show that f has both absolute maximum and absolute minimum.

$*$ Enough to prove f is Continuous at 0 :

$\boxed{x \neq 0}$

$$-x < x \sin\left(\tfrac{1}{x}\right) < x$$

$$\downarrow \qquad\qquad \downarrow$$

$$0 \qquad\qquad 0$$

$\therefore$ By Squeezing Thr. $\boxed{\lim\limits_{x \to 0} x \sin\left(\frac{1}{x}\right) = 0 = f(0)}$

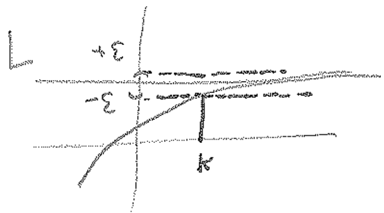
$\therefore f$ is contin on $[-2, 2]$ "closed + Bd interval"

Then by part (a) f has absolute max and abs. min.

[Q.4, 5 pts](a) Let $f : (0, \infty) \to \mathbb{R}$ be a function. Show that

two parts Proof

$$\lim_{x\to 0+} f\left(\frac{1}{x}\right) = L \quad \boxed{\text{if and only if}} \quad \lim_{x\to\infty} f(x) = L$$



First part $(\Rightarrow)$ Suppose $\lim\limits_{x\to 0+} f(\frac{1}{x}) = L.$

Let $\varepsilon > 0$. Want $K(\varepsilon)$ s.t $\forall\ x > K(\varepsilon)$; $|f(x) - L| < \varepsilon$.

as given, $\exists\ \delta > 0$ s.t

one-side limit: $\forall x;\ 0 < x < \delta \implies \left|f\left(\frac{1}{x}\right) - L\right| < \varepsilon.$ ---- $(*)$

choose $\boxed{K := \frac{1}{\delta}}$

Now, if $x > K \implies 0 < \frac{1}{x} < \frac{1}{K} = \delta$

By $(*)$ $\implies \left|f\left(\frac{1}{\frac{1}{x}}\right) - L\right| = |f(x) - L| < \varepsilon.$

Second part $(\Leftarrow)$ Suppose $\lim\limits_{x\to\infty} f(x) = L.$

Let $\varepsilon > 0$. Want $\delta > 0$ s.t

$\forall x;\ 0 < x < \delta \implies \left|f\left(\frac{1}{x}\right) - L\right| < \varepsilon.$

As given, $\exists\ K$ s.t

$\forall x > K\ ;\ |f(x) - L| < \varepsilon.$ ---- $(**)$

choose $\boxed{\delta = \frac{1}{K}} > 0$. Then

$\forall x;\ 0 < x < \delta \implies \frac{1}{x} > \frac{1}{\delta} = K$

Then by $(**)$; $\left|f\left(\frac{1}{x}\right) - L\right| < \varepsilon.$

// Method (2) By sequences:

$\exists\ x_n;\ x_n \longrightarrow c;\ x_n \neq c\ \forall n,$

f is cont. $\Rightarrow f(x_n) \longrightarrow f(c) \notin f(A)$

$f(x_n) \in f(A);\ f(x_n) \neq f(c)\ \therefore f(x_n)$ is cluster of $f(A)$.

[Q.5, 4 pts] Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. Show that if c is a cluster point of $A \subseteq \mathbb{R}$, and $f(c) \notin f(A)$, then $f(c)$ ia a cluster point of $f(A)$.
Recall that $f(A) = \{f(a),\ \forall a \in A\}$.

Method (1) Proof: Let $U_\varepsilon(f(c))$ be any ε-nhold of $f(c)$.

Since f is continuous at c, then

$\exists$ a δ-nhold $V_\delta(c)$ s.t.

$\boxed{\forall x \in V_\delta(c) \Longrightarrow f(x) \in U_\varepsilon(f(c))}$. ---- $(*)$

Since c is a cluster point of A;

then $\exists\ a \in A;\ a \neq c$ s.t.

$a \in V_\delta(c)$.

$\therefore$ By $(*)$, $f(a) \in U_\varepsilon(f(c))$

indeed, $f(a) \neq f(c)$ since $f(c) \notin f(A)$.

$\therefore f(c)$ is a cluster point of $f(A)$. $\blacksquare$

Good Luck

please, look here....

Idea of proof