United Arab Emirates University

Department of Mathematical Sciences

Number Theory, Section 52

Time: 120 minutes | Final Exam | June 8, 2010

Textbooks or notes may **not** be used.

Name:

ID:

Show all your work

1. **(6 Points)**
   Use mathematical induction to prove that for all $n \geq 1$:
   (a) $\binom{2}{2} + \binom{3}{2} + \binom{4}{2} + \ldots + \binom{n}{2} = \binom{n+1}{3}$.

   (b) $64|(3^{2n+3} + 40n - 27)$.

2. **(6 Points)**
(a) Use the Euclidean algorithm to find $\gcd(2378, 1769)$.

(b) Express $\gcd(2378, 1769)$ as a linear combination of 2378 and 1769.

3. **(6 Points)**

(a) Solve the Diophantine equation $858x + 253y = 33$.

(b) Solve the following system of congruences:

$$\begin{cases} 11x + 5y \equiv 7 \pmod{20} \\ 6x + 3y \equiv 8 \pmod{20} \end{cases}$$

4. **(6 Points)**
(a) Find the remainder when $375 \cdot 2^{100} - 35^{87}$ is divided by 6.

(b) If the remainder is 5 when n is divided by 8, find the remainder when $n^3 + 5n$ is divided by 8.

5. **(6 Points)**
(a) Without performing the division, determine whether the integer $149,235,678$ is divisible by 9 or 11.

(b) If a is an **odd** integer, prove that for any integer $n \geq 1$ we have $a^{2^n} \equiv 1 \pmod{2^{n+2}}$.
(**Hint:** use induction on n.)

6. **(6 Points)**
(a) Solve the quadratic congruence $x^2 \equiv -1 \pmod{23}$.

(b) If p is a prime of the form $4k+3$, prove that

$$2 \cdot 4 \cdot 6 \cdots (p-1) \equiv \pm 1 \pmod{p}.$$

(**Hint:** $(\frac{p-1}{2})! \equiv \pm 1 \pmod{p}$ and $2^{\frac{p-1}{2}} \equiv \pm 1 \pmod{p}$.)

7. **(4 Points)**
(a) Find $\tau(756)$ and $\sigma(756)$.

(b) Find $\phi(324)$ and
$$\sum_{\substack{1\leq k<324\\ \gcd(k,324)=1}} k.$$