United Arab Emirates University

Department of Mathematical Sciences

Number Theory, Section 01

Time: 120 minutes | Final Exam | June 5, 2010

Textbooks or notes may **not** be used.

Name:

ID:

Section:

Instructor:

Show all your work.

1. **(5 Points)**
   Use mathematical induction to prove that for all $n \geq 1$:
   (a) $1^2 + 3^2 + 5^2 + \ldots + (2n-1)^2 = \binom{2n+1}{3}$.

   (b) $5 | 3^{3n+1} + 2^{n+1}$.

2. **(5 Points)**
(a) Use the Euclidean algorithm to find $\gcd(4928, 1771)$.

(b) Express $\gcd(4928, 1771)$ as a linear combination of 4928 and 1771.

3. **(5 Points)**
Determine all integer solutions of the Diophantine equation

$$123x + 360y = 99.$$

4. **(5 Points)**
Use the theory of congruences to:
(a) Find the remainder when 41^{65} is divided by 7.

(b) Show that $7|5^{2n} + 3 \cdot 2^{5n-2}$ for all $n \geq 1$.

5. **(5 Points)**
Solve the following system of congruences:

$$\begin{cases} x \equiv 5 \pmod{6} \\ x \equiv 4 \pmod{11} \\ x \equiv 3 \pmod{17} \end{cases}$$

6. **(6 Points)**
Let $p > 2$ be a prime number and let a and b be integers not divisible by p.
(a) If $a^p \equiv b^p \pmod{p}$, prove that $a \equiv b \pmod{p}$.

(b) If $k \in \mathbb{Z}$, prove that p^2 divides $(b + pk)^p - b^p$.

(c) If $a^p \equiv b^p \pmod{p}$, use parts (a) and (b) to prove that $a^p \equiv b^p \pmod{p^2}$.

7. **(4 Points)**

(2) Solve the quadratic congruence

$$x^2 + 1 \equiv 0 \pmod{23}.$$

(b) Find the remainder when $2(26!)$ is divided by 29.

8. **(5 Points)**
(a) Find $\sigma(756)$ and $\tau(324)$.

(b) If p and $2(2p-1)$ are both odd primes, prove that $\Phi(n+2) = \Phi(n)$.