Differential Equations and Engineering Applications

MATH 2210

Applications of Laplace Transform

Introduction to Laplace Transform (1/3)

- **For any function f(t), its Laplace Transform is given as:**

$$F(s) = \int_{0}^{\infty} e^{-st} f(t)dt$$

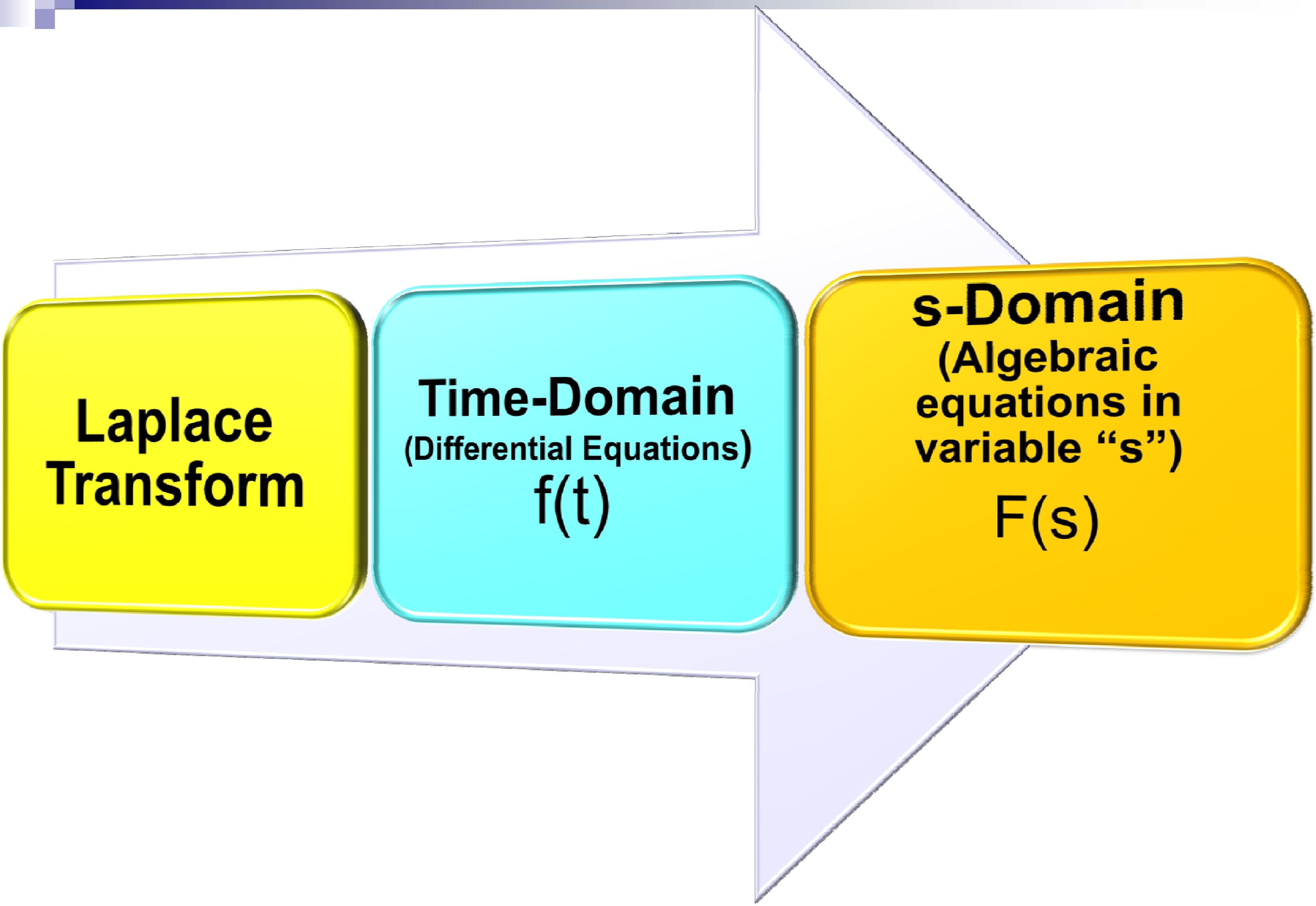
- f(t) is said to be in time-domain, where the independent variable, time t is a real quantity
- F(s) [Laplace Transform of f(t)] is said to be in s-domain, where independent variable, s is a complex quantity

Introduction to Laplace Transform (2/3)

- **Laplace Transform is a mathematical tool, which helps in analyzing the system of interest.**

▪It takes us from Time-Domain description of system (represented by Differential Equations) to a s-Domain description (represented by algebraic equations in variable "s").

▪It is relatively easy to solve algebraic equation, rather than the differential equations.


Laplace Transform
Time-Domain
(Differential Equations)
f(t)
s-Domain
(Algebraic equations in variable "s")
F(s)

Introduction to Laplace Transform (3/3)

- **Solution of algebraic equations in "s", with zero initial conditions, gives Transfer Function of the system.**
  - **Transfer Function = <u>Ratio of the Laplace Transform of the output to the Laplace Transform of input,</u> when the initial conditions are zero.**
  - **Once we know the transfer function of a system, we can easily evaluate the response of the system for any input signal.**
  - **<u>The roots of denominator of the transfer function</u> (called as Poles of the system) determine the stability of system.**
  - **System is STABLE if all poles have –ve real part.**

Properties of Laplace Transform

$f(t)$	$F(s)$
$af(t)+bg(t)$	$aF(s)+bG(s)$
$f'(t)$	$sF(s)-f(0)$
$f''(t)$	$s^2F(s)-sf(0)-f'(0)$
$\int_0^t f(\tau)d\tau$	$\frac{1}{s}F(s)$

Properties of Laplace Transform

$t^n f(t)$	$(-1)^n \frac{d^n F(s)}{ds^n}$
$\frac{1}{t} f(t)$	$\int_s^\infty F(\sigma) d\sigma$
$e^{at} f(t)$	$F(s-a)$

Some Useful Laplace Transform Pairs

$f(t)$	$F(s)$
1	$\frac{1}{s}$
t	$\frac{1}{s^2}$
e^{at}	$\frac{1}{s-a}$
$\sin(at)$	$\frac{a}{s^2+a^2}$
$\cos(at)$	$\frac{s}{s^2+a^2}$

Application of Laplace Transform to Differential Equations

- **Consider 1st order differential equation:**

$$y'(t) + ay(t) = x(t); \quad y(0)$$

- **Taking Laplace Transform of both sides, this time-domain equation can be written in s-domain as**

$$[sY(s) - y(0)] + a[Y(s)] = X(s)$$

$$\Rightarrow Y(s) = \underbrace{\left(\frac{1}{s+a}\right)X(s)}_{\text{Forced Response}} + \underbrace{\left(\frac{1}{s+a}\right)y(0)}_{\text{Natural Response}}$$

Application of Laplace Transform to Differential Equations

- **The response of system to the forcing function alone is called forced response:**

$$Y_{(f)}(s) = \left(\frac{1}{s+a}\right) X(s) \Rightarrow y_f(t)$$

- **The response of system due to the initial conditions of systems only is called natural response:**

$$Y_{(n)}(s) = \left(\frac{1}{s+a}\right) y(0) \Rightarrow y_n(t)$$

- **The total response of the system:**

$$y(t) = y_f(t) + y_n(t)$$

Application of Laplace Transform to Differential Equations

- The transfer function of the system is defined as:

Transfer Function

OUTPUT(displacement, temperature,…)

$$H(s)=\left.\frac{Y(s)}{X(s)}\right|_{\text{I.Cs=zero}}=\frac{N(s)}{D(s)}=\left(\frac{1}{s+a}\right)$$

Numerator

Denominator

INPUT(force, heat,…)

where

- N(s) and D(s) are polynomials in "s".
- The roots of N(s) are called ZEROs of system.
- The roots of D(s) are called POLEs of the system.

Application of Laplace Transform to Differential Equations

Transfer Function Analysis:

- **Response of System:** Once transfer function H(s) of the system is known to us, the response of system for any input can be calculated as:

OUTPUT | Transfer Function | INPUT | Response

$$Y(s) = H(s)X(s), \quad \text{and} \quad y(t) = L^{-1}\{Y(s)\}$$

- **Stability of System:** The stability of the system can be checked from the poles of H(s), i.e., the roots of the denominator polynomial D(s), as

System is STABLE if all poles have –ve real part.

Activity:

1)Find the Laplace transformation of the following equation:

$$m\frac{dx^2}{dt^2}+B\frac{dx}{dt}+kx=f(t)$$

2)What are the Force and the Natural response of the system

3)Find Transfer Function

Activity:

1)For the following TF, find zeros and the stability of the system:

$$H(s) = \frac{s+1}{s^2 + 5s + 6}$$

2)Find the corresponding DE?

s-Domain Analysis in MATLAB

Transfer Function H(s)Representation

Using MATLAB:

$$H(s) = \frac{N(s)}{D(s)} = \frac{s+2}{s^2+2s+10}$$

1) DEFINE NUMERATOR ⇨ **num=[**no1, no2, …, …]

2) DEFINE DENOMINATOR ⇨ **den=[**no1, no2, …, …]

3) DEFINE TRANSFER FUNCTION ⇨ **sys=tf(num,den)**

Just name

Transfer Function Generator Command

```
>> % Coefficients of N(s)
>> num=[1 2];
>> % Coefficients of D(s)
>> den=[1 2 10];
>> % Transfer Function H(s) of system
>> sys=tf(num,den);
>> % Type sys and check the resutls displayed
>>sys
```

$$\frac{s+2}{s^2+2s+10}$$

```
Transfer function:
    s + 2
--------------
s^2 + 2 s + 10
```

OUTPUT

1)Response of System (*Zero Initial Conditions*

Impulse Load

Find **H(s)** by defining:

num and **den** using:

tf(num,den)

impulse(sys,time)

Impulse Command computes the ***impulse response*** of a function

Laplace Function

```
[f,t] = impulse(sys,t);
where t = ti:dt:tf
```

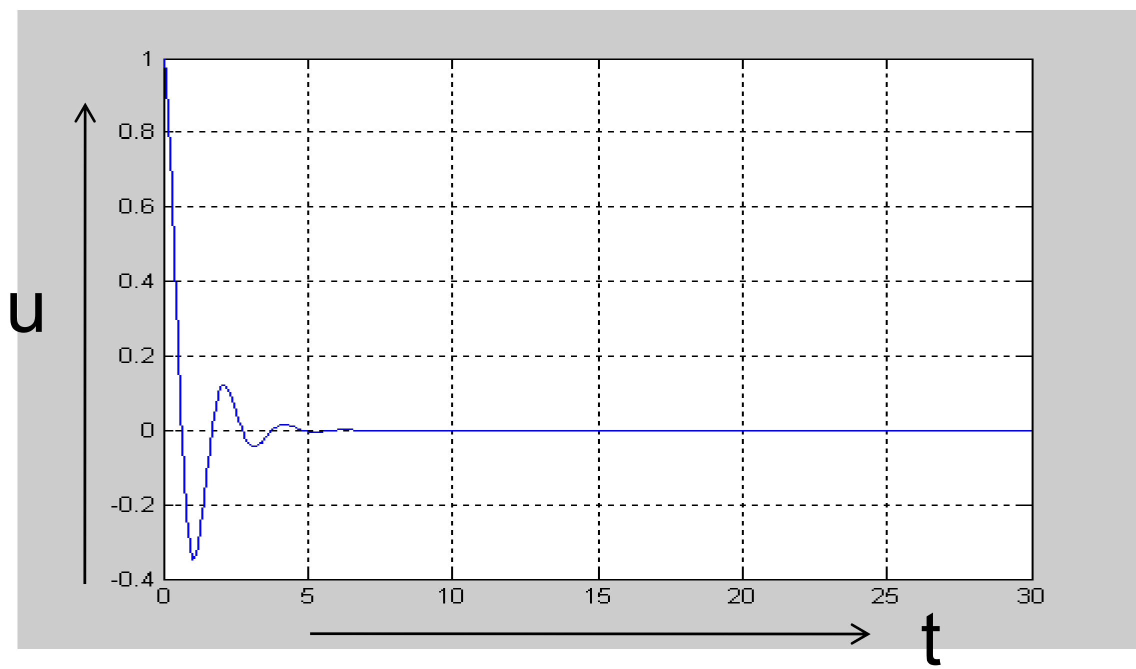
Time

Example: **Find impulse response of the following function for 0<t<30 :**

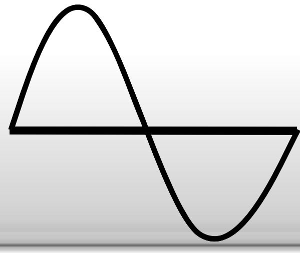
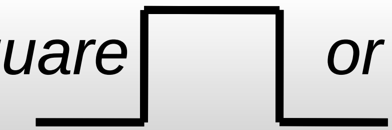
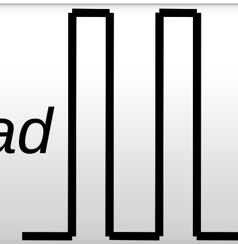
$$H(s) = \frac{N(s)}{D(s)} = \frac{s+2}{s^2+2s+10}$$

$$H(s) = \frac{N(s)}{D(s)} = \frac{s+2}{s^2+2s+10}$$

```
>> num=[1 2];        % Coefficients of N(s)
>> den=[1 2 10]; % Coefficients of D(s)
>> sys=tf(num,den); % System Trans.Func. H(s)
>> t1=0:0.1:30;      % Time vector
>> [h,t] = impulse(sys,t1);
>> plot(t,h), grid
```



2)Response of System (*Zero Initial Conditions*)

Sine  *or Square*  *or Pulse Load* 

Find **H(s)** by defining
num and **den** using:
tf(num,den)

Generate signal type using:
[x,t]=gensig('type',period,duration,step)

Solve system by:
lsim(sys,x,time)

s-Domain Analysis in MATLAB

Generating Signals: The MATLAB command `gensig` is used to generate time-domain analog signals.

[u,t] = `gensig` ('Type',tau,tf,ts)

Generate Signal

Type = sine, square, pulse
tau = Period (one cycle)
tf = Duration (total time)
ts = Time spacing (step)
u = value of signal
t = corresponding time instants

s-Domain Analysis in MATLAB

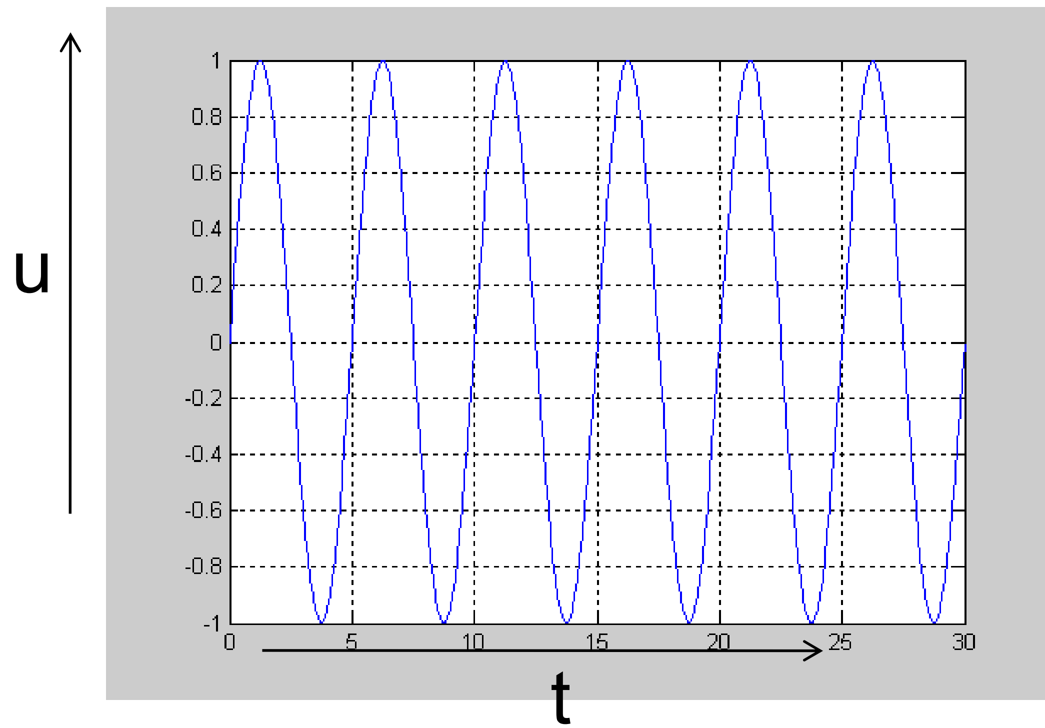
```
>> % Generate and Plot Sine wave of period 5 sec for
   a total time of 30 sec.

>> [u,t] = gensig('sine',5,30,0.1);
>> plot(t,u), grid
```

Total Time

Type

Period

Step



$$if\ \ H(s)\ \text{is given and it requeste to find}\ \ y(t)\ \text{for}\ x(t)$$

Transfer Function (known)

OUTPUT(Desired) ?

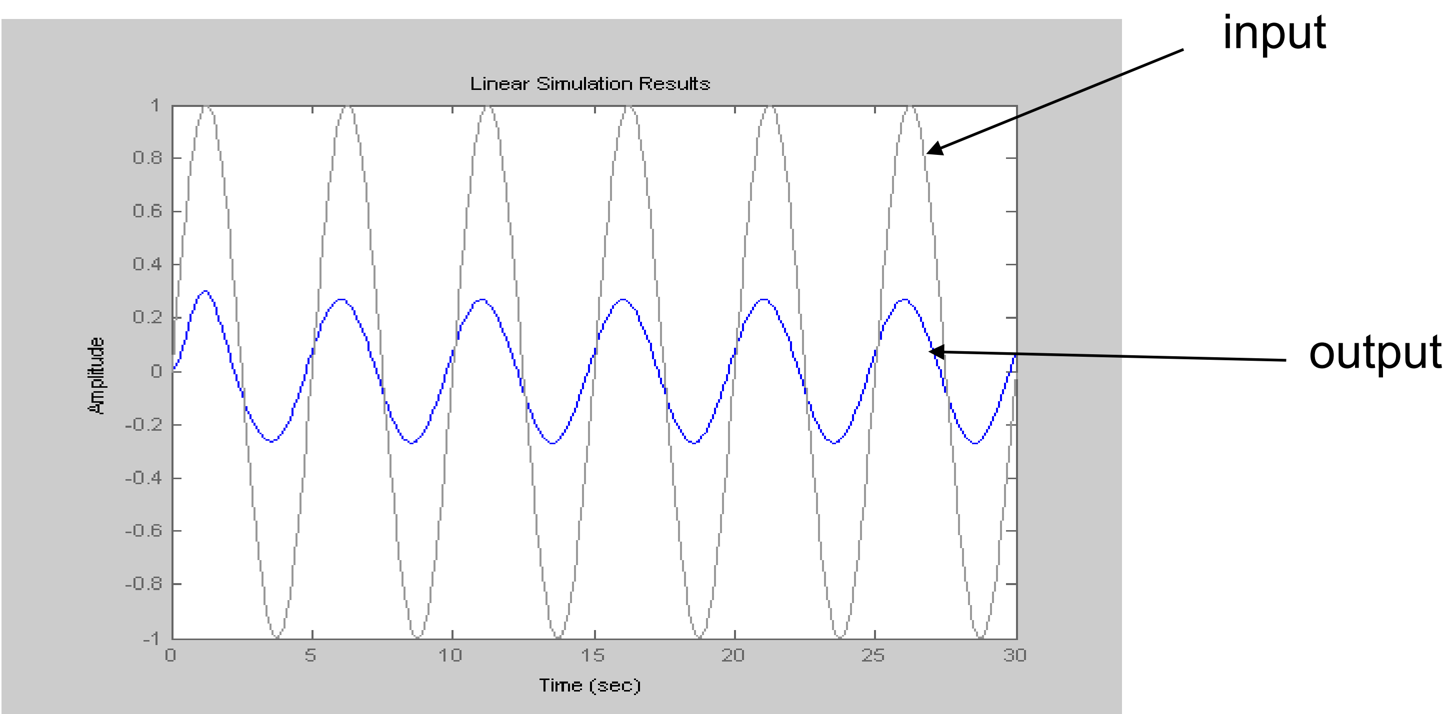
$$H(s)=\left.\frac{Y(s)}{X(s)}\right|_{\text{I.Cs=zero}}$$

INPUT (available)

1) Generate the input signal `x(t) using gensig`
2) `Generate Transfer Function` sys=tf(num,den)
3) `Use lsim(sys,x,t)`

Ex.:Find response of the following function $H(s) = \frac{N(s)}{D(s)} = \frac{s+2}{s^2+2s+10}$

```
>> % System Transfer Function H(s)
>> num=[1,2];  den=[1,2,10];
>> sys=tf(num,den);
>> [x,t] = gensig('sine',5,30,0.1);
>> lsim (sys,x,t);
```



Initial Conditions are not zero

Find **H(s)** by defining:

num and **den** using:

sys=tf(num,den)

Find state space by using:

[A,B,C,D]=tf2ss(num,den)

Generate signal type using:

[x,t]=gensig('type',period,duration,step)

Solve system by:

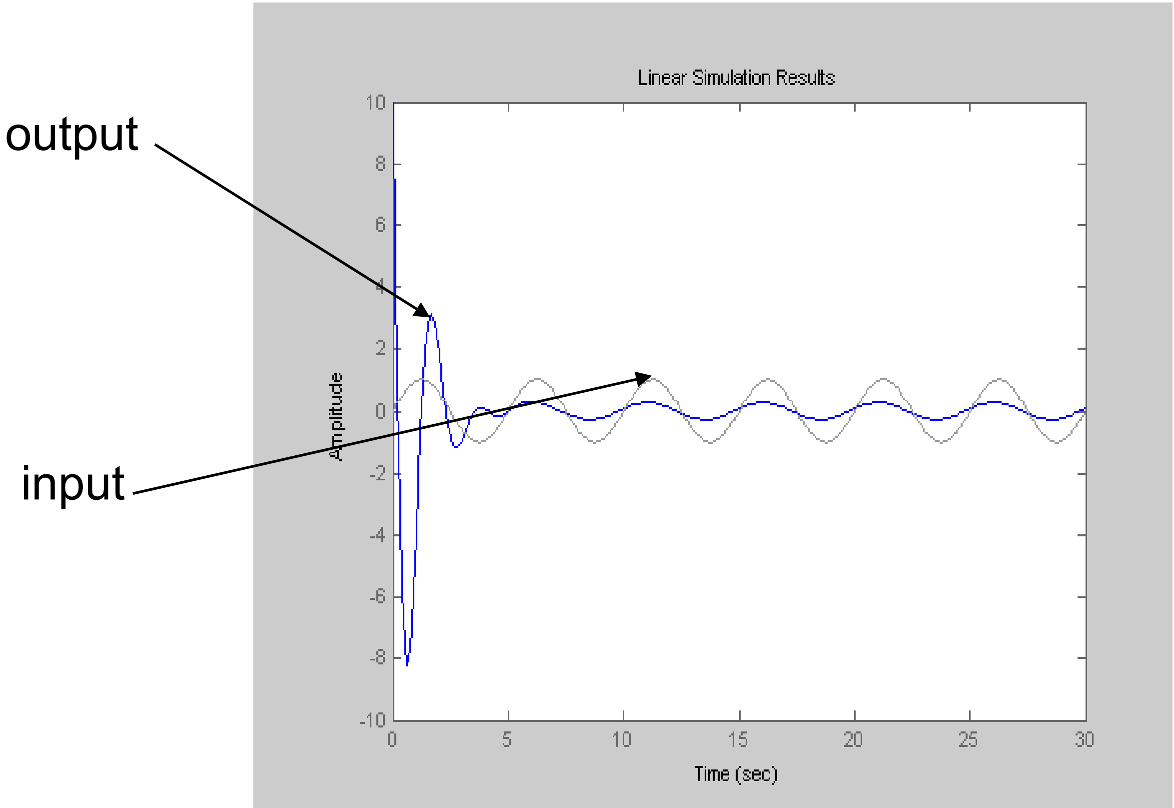
lsim(A,B,C,D,x,time,IC)

Response of System (Non-zero Initial Conditions):

$H(s)$ is given and it is requested to find $y(t)$ for $x(t)$

- Generate the input signal x(t)
- Represent the system in State- space form using
- `tf2ss(num,den)`
- `lsim(A,B,C,D,x,t,x0);` where sys is system description is state-space and x0 is initial state vector

```
>> % System Transfer Function H(s)
>> num=[1,2]; den=[1,2,10];
>> % Transfer Function to State Space
>> [A,B,C,D]=tf2ss(num,den);
>> % Generate the input signal x(t)
>> [x,t] = gensig('sine',5,30,0.1);
>> lsim(A,B,C,D,x,t,[0 5]);
```



Applications of Laplace Transform

1. Thermal Systems

Variables:

- **Temperature: θ [kelvin]]**
- **Heat flow rate, q [J/s]**

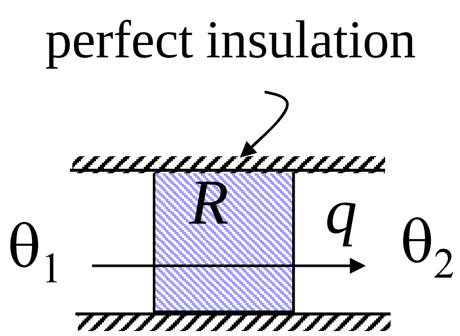
Element Laws:

- Thermal Capacitance: $$\dot{\theta}(t) = \frac{1}{C}\left[q_{in}(t) - q_{out}(t)\right],$$

where C: thermal capacitance [J/K]

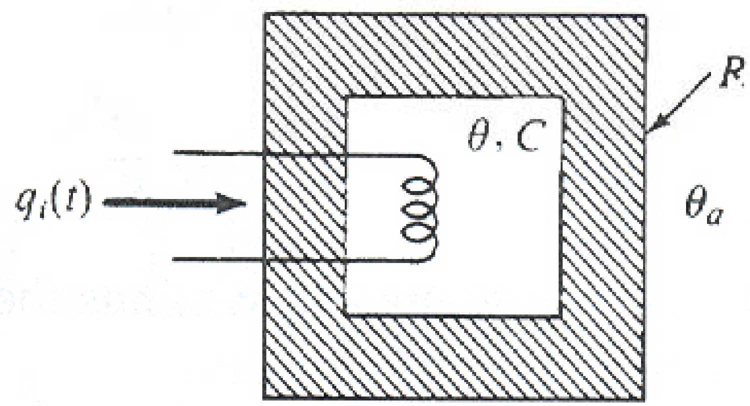
- Thermal Resistance: $$q(t) = \frac{1}{R}\left[\theta_1(t) - \theta_2(t)\right],$$

where R: thermal resistance of the path between the two bodies [Ks/J]



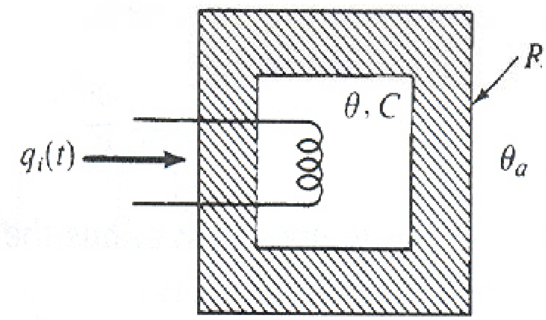
Example 1:

A thermal capacitance C, is enclosed by insulation of resistance R. Heat is added at a rate $q_i(t)$. The ambient temperature surrounding the exterior is Θ_a. Find the system model in terms of $\Theta(t)$, $q_i(t)$, and Θ_a.



Example 1:

Differential Equation:



$$\dot{\theta}(t) = \frac{1}{C}\left[q_{in}(t) - q_{out}(t)\right],$$

$$where \;\; q_{in}(t) = q_i(t),$$

$$q_{out}(t) = \frac{1}{R}\left(\theta(t) - \theta_a\right).$$

$$Hence, \;\; \dot{\theta}(t) = \frac{1}{C}\left[q_i(t) - \frac{1}{R}\left(\theta(t) - \theta_a\right)\right],$$

$$\boxed{\dot{\theta}(t) + \frac{1}{RC}\theta(t) = \frac{1}{C}q_i(t) + \frac{1}{RC}\theta_a.}$$

Example 1:
s-Domain Representation

- **Taking Laplace Transform of the differential equation:**

$$s\theta(s) - \theta(0) + \frac{1}{RC}\theta(s) = \frac{q_i}{sC} + \frac{\theta_a}{sRC}$$

$$\Rightarrow \left[s + \frac{1}{RC}\right]\theta(s) = \left[\frac{q_i}{sC} + \frac{\theta_a}{sRC}\right] + \theta(0)$$

$$\Rightarrow \theta(s) = \frac{\left[\frac{q_i}{C} + \frac{\theta_a}{RC}\right]}{s\left[s + \frac{1}{RC}\right]} + \frac{\theta(0)}{\left[s + \frac{1}{RC}\right]} = \frac{\theta(0)s + \left[\frac{q_i}{C} + \frac{\theta_a}{RC}\right]}{s\left[s + \frac{1}{RC}\right]}$$

$$\text{and} \quad \theta(t) = L^{-1}\{\theta(s)\}$$

Example 1:
Simulation Parameters

Simulate the thermal capacitance system for:

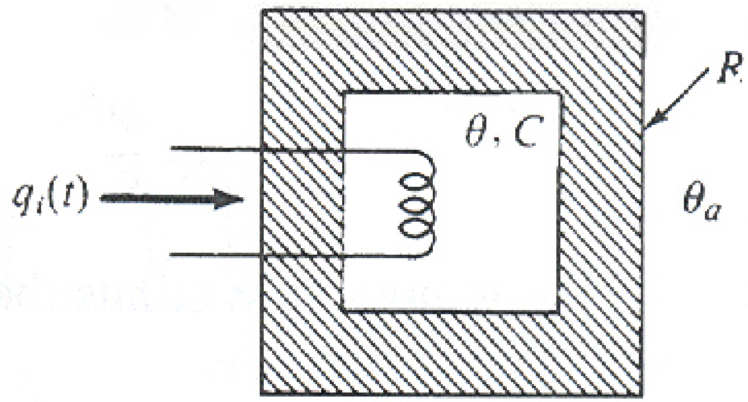
$C=1.0*10^3$ J/K,

$R=2.0*10^{-3}$ s-K/J,

$\Theta_a=300$ K, and

$q_i(t)=1000$ K.

Assume $\Theta(0)=\Theta_a=300$ K.



Example 1:
Function for ODE Solver

```
function dthetadt = ThermalEx1(t,theta)
R=2e-3;
C=1e3;
qi=1000;
theta_a=300;
dthetadt=[(1/(R*C))*theta(1)+(1/C)*qi+(1/(R*C))*theta_a];
```

Example 1: MATLAB Simulation

$$s\theta(s)-\theta(0)+\frac{1}{RC}\theta(s)=\frac{q_i}{sC}+\frac{\theta_a}{sRC}$$

$$\Rightarrow\left[s+\frac{1}{RC}\right]\theta(s)=\left[\frac{q_i}{sC}+\frac{\theta_a}{sRC}\right]+\theta(0)$$

$$\Rightarrow\theta(s)=\frac{\left[\frac{q_i}{C}+\frac{\theta_a}{RC}\right]}{s\left[s+\frac{1}{RC}\right]}+\frac{\theta(0)}{\left[s+\frac{1}{RC}\right]}=\frac{\theta(0)s+\left[\frac{q_i}{C}+\frac{\theta_a}{RC}\right]}{s\left[s+\frac{1}{RC}\right]}$$

ODE analysis

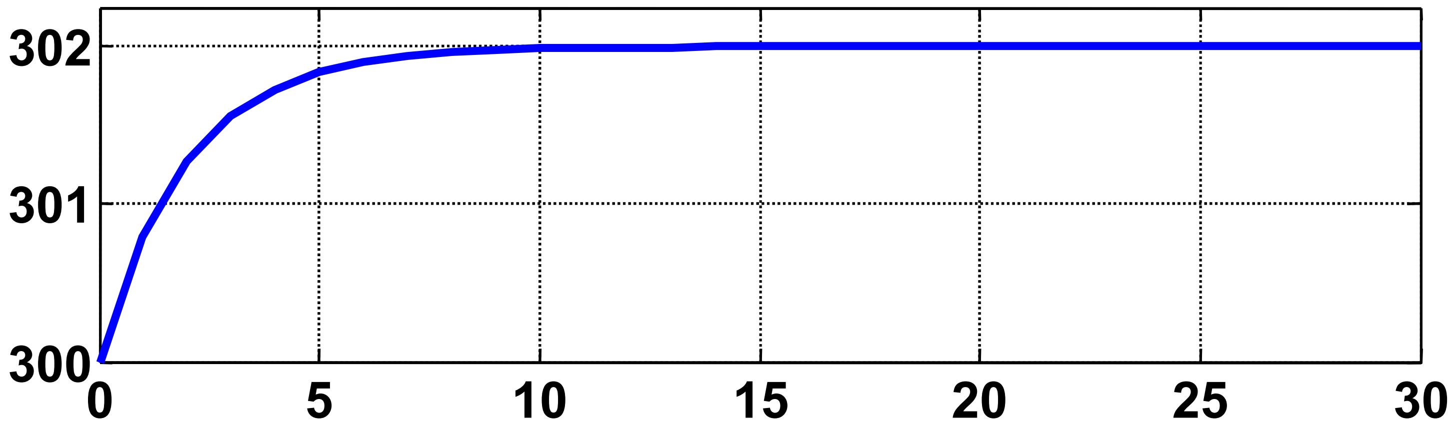
```
>> clear all, clc
>> t=0:30;
>> [t1,theta_ode] = ode45
```

transfer function A

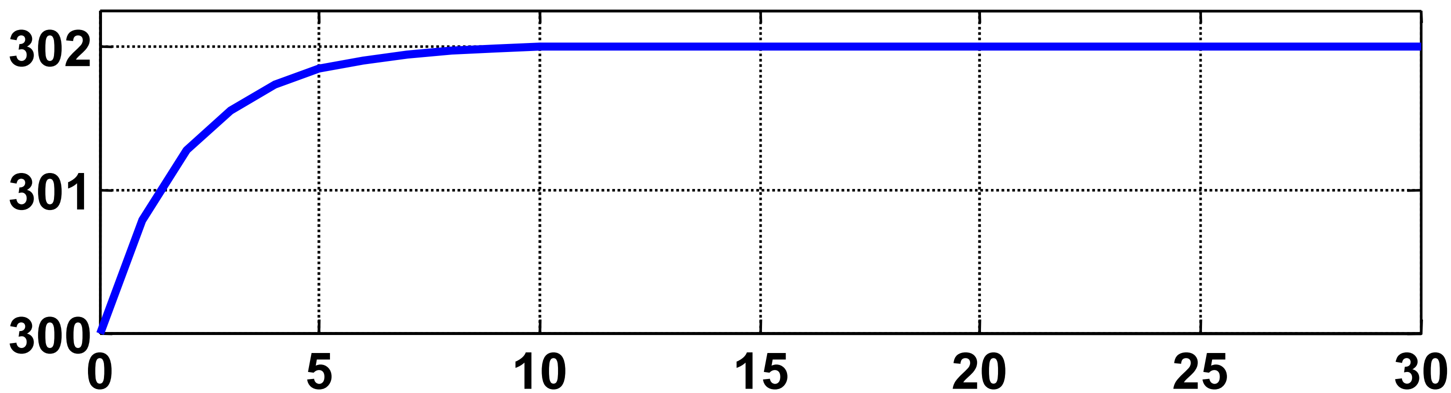
```
>> R=2e-3;  C=1e3; qi=1000;
>> theta_a=300; theta_zero=300;
>> num = [theta_zero ,((1/C)*qi+(1/(R*C))*theta_a)];
>> den = [1, 1/(R*C), 0];
>> sys=tf(num,den);
>> [theta_tf, t2]=impulse(sys, t);
>> subplot(2,1,1), plot(t1,theta_ode), title('ODE Analysis')
>> subplot(2,1,2), plot(t2,theta_tf),
>> title('Transfer Function Analysis')
```

Example 1: MATLAB Simulation Results

ODE Analysis

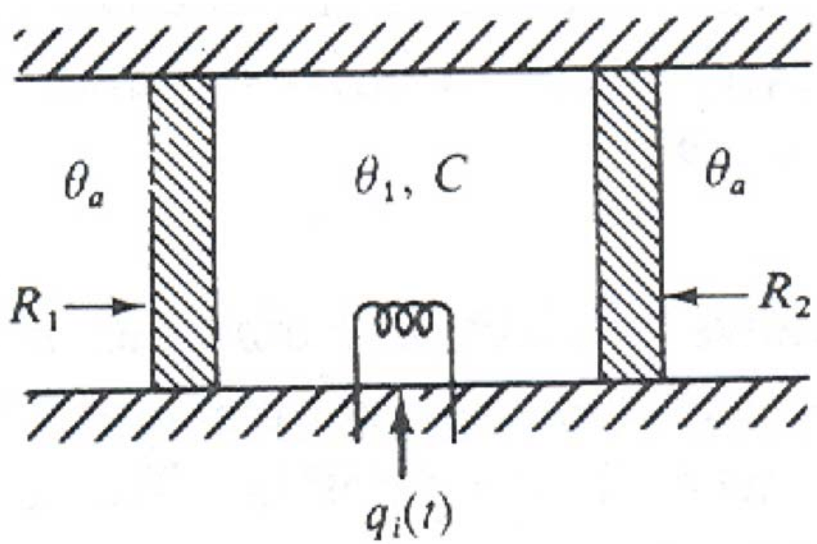


Transfer Function Analysis



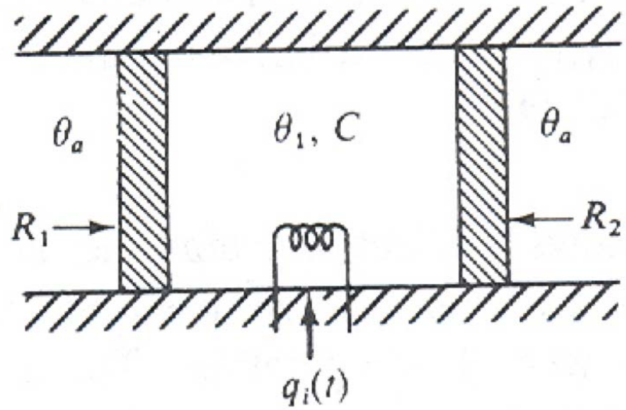
Example 2:

Consider the following thermal capacitance, C1, of temperature ϴ1(t). It is assumed that the system is perfectly insulated except for the thermal resistances R1 and R2. Heat is added at a rate qi(t), and the ambient temperature is ϴa.



Example 2:

Differential Equation:



$$\dot{\theta}_1(t) = \frac{1}{C1}\left[q_{in}(t) - q_{out}(t)\right],$$

$$where \quad q_{in}(t) = q_i(t),$$

$$q_{out}(t) = \frac{1}{R1}\left(\theta_1(t) - \theta_a\right) + \frac{1}{R2}\left(\theta_1(t) - \theta_a\right)$$

$$Hence, \;\; \dot{\theta}_1(t) = \frac{1}{C1}q_i(t) - \frac{1}{C1}\left(\frac{1}{R1} + \frac{1}{R2}\right)\theta(t) + \frac{1}{C1}\left(\frac{1}{R1} + \frac{1}{R2}\right)\theta_a,$$

$$\dot{\theta}(t) + b\theta(t) = aq_i(t) + b\theta_a, \;\; b = \frac{1}{C1}, \;\; a = \frac{1}{C1}\left(\frac{1}{R1} + \frac{1}{R2}\right).$$

Example 2:

s-Domain Representation

- **Taking Laplace Transform of the differential equation:**

$$s\theta(s) - \theta(0) + b\theta(s) = \frac{aq_i}{s} + \frac{b\theta_a}{s}$$

$$\Rightarrow [s+b]\theta(s) = \left[\frac{aq_i}{s} + \frac{b\theta_a}{s}\right] + \theta(0)$$

$$\Rightarrow$$

$$\theta(s) = \frac{\theta(0)s + [aq_i + b\theta_a]}{s[s+b]}$$

$$\text{and} \quad \theta(t) = L^{-1}\{\theta(s)\}$$

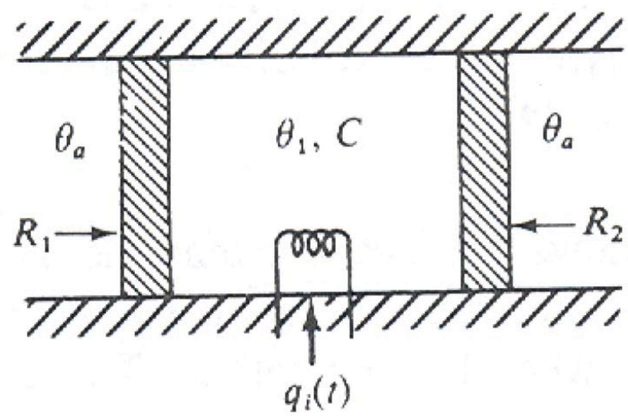
Example 2:
Simulation Parameters

Simulate the system for 5 sec.

$C1=1.0*10^3$ J/K,

$R1=2.0*10^{-3}$ s-K/J,

$R2=1.5*10^{-3}$ s-K/J,



$\Theta_a=300$ K, and

$q_i(t)=1000$ K.

Assume $\Theta(0)=\Theta_a=300$ K.

Example 2:
Function for ODE Solver

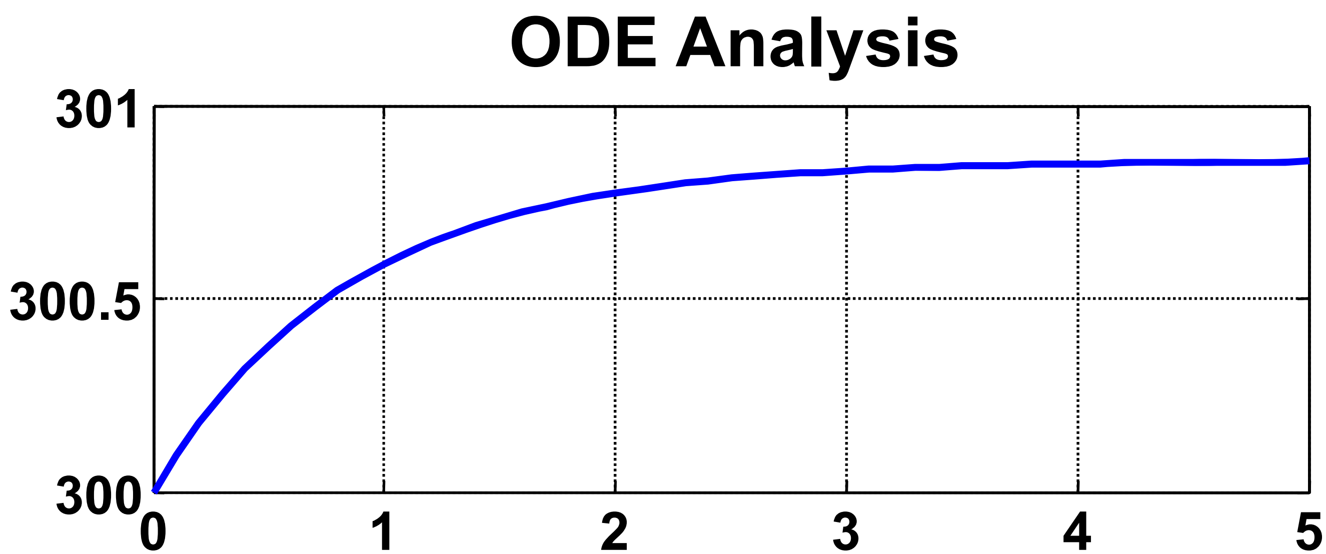
```
function dthetadt = ThermalEx2(t,theta)
C1=1e3;
R1=2e-3;
R2=1.5e-3;
a=1/C1;
b=1/C1*(1/R1+1/R2);
qi=1000;
theta_a=300;
dthetadt = [-b*theta(1)+a*qi+b*theta_a];
```

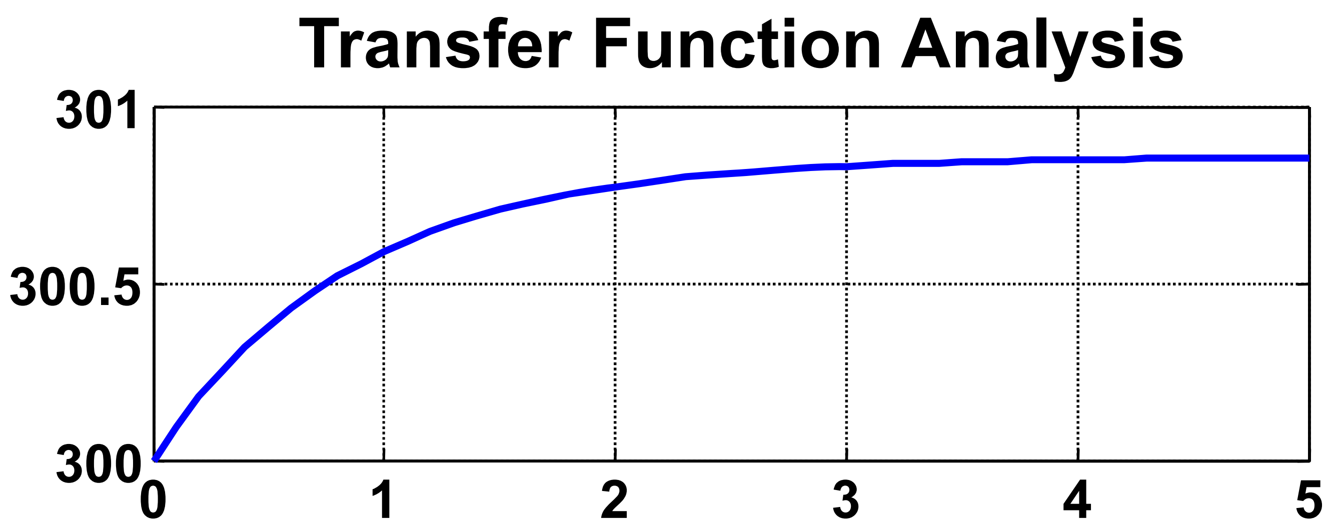
Example 2: MATLAB Simulation

```
clear all, clc

t=0:0.1:5;
ODE analysis
[t1,theta_ode] = ode45('ThermalEx2',t,[300]);

Transfer function Analysis
R1=2e-3; R2=1.5e-3; C1=1e3; qi=1000;
a=1/C1; b=1/C1*(1/R1+1/R2);
theta_a=300; theta_zero=300;
num = [theta_zero,(a*qi+b*theta_a)];
den = [1,b,0];
sys=tf(num,den);
[theta_tf, t2]=impulse(sys,t);

subplot(2,1,1), plot(t1,theta_ode), title('ODE Analysis')
subplot(2,1,2), plot(t2,theta_tf),
title('Transfer Function Analysis')
```

$$\theta(s) = \frac{\theta(0)s + \left[aq_i + b\theta_a\right]}{s\left[s+b\right]}$$

Example 2: MATLAB Simulation Results





Applications of Laplace Transform

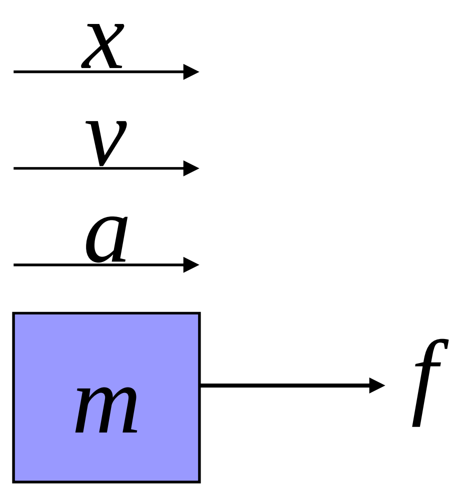
2. Mechanical Systems

Variables:

- Displacement x
- Velocity v=dx/dt
- Acceleration a=dv/dt

Element Laws:

- **Mass** (Newton's second law):



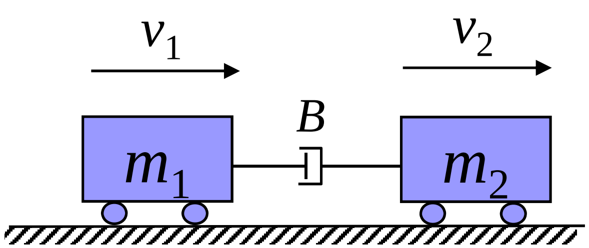
$$f = \frac{d}{dt}(mv)$$

$$f = \frac{d}{dt}(mv) = m\frac{d}{dt}v = ma = m\ddot{x}$$

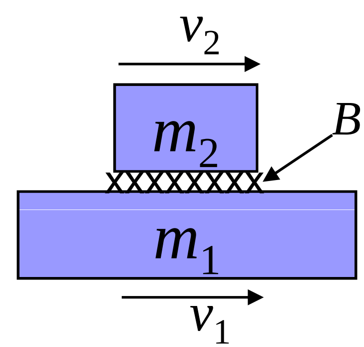
Element Laws:

- **Friction** (Friction resistance force):

$$f_f = B\Delta v = B(v_2 - v_1)$$



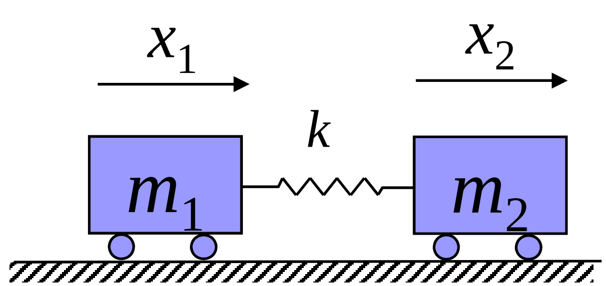
dashpot representation



Oil film representation

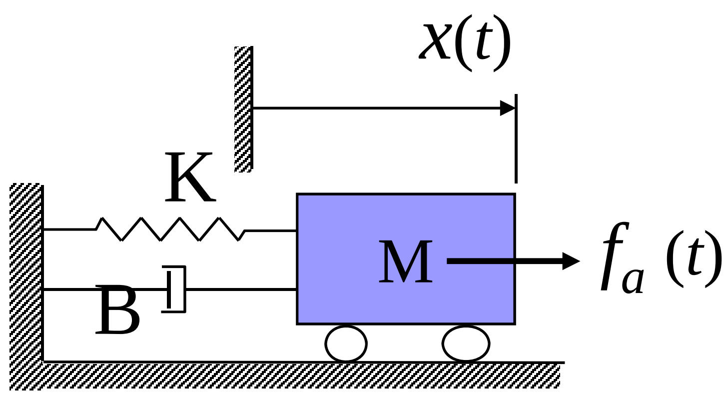
- **Stiffness** (Stiffness resistance force):

$$f_s = k\Delta x = k(x_2 - x_1)$$



Example 1

Consider the following model

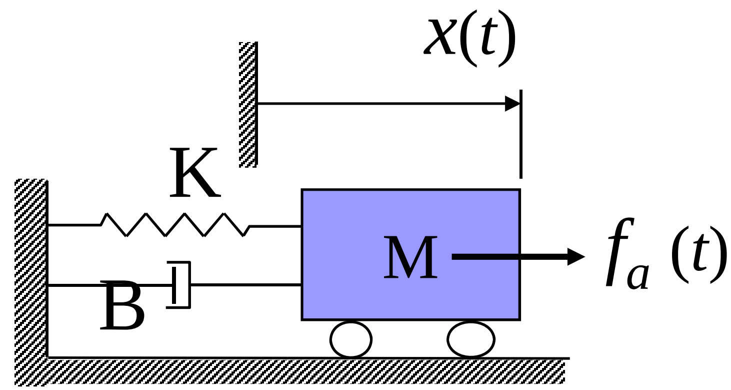


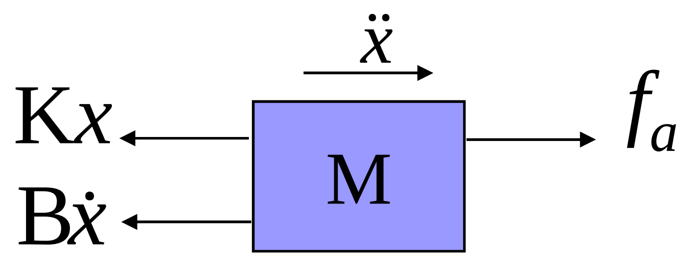
M=1, K=100, B=2

- Obtain differential equation representation.
- Obtain s-domain representation
- Simulate the system for following cases:
  - CASE 1: Initial Conditions are zero, and $f_a(t)=10$
  - CASE 2: Initial Conditions are zero, and $f_a(t)= 50\sin(2t)$
  - CASE 3: I.Cs. x=[2.5; 2.5],and $f_a(t)= 50\sin(2t)$

Example 1 - Differential Equation

$$\ddot{x} + \frac{B}{M}\dot{x} + \frac{K}{M}x = \frac{1}{M}f_a$$





Example 1 - s-Domain Representation

$$X(s) = \frac{(\frac{1}{M})F_a(s)}{(s^2 + \frac{B}{M}s + \frac{K}{M})} + \frac{(s + \frac{B}{M})x(0) + \dot{x}(0)}{(s^2 + \frac{B}{M}s + \frac{K}{M})}$$

$$\Rightarrow$$

$$X^{(f)}(s) = \frac{(\frac{1}{M})F_a(s)}{(s^2 + \frac{B}{M}s + \frac{K}{M})}, \quad X^{(n)}(s) = \frac{(s + \frac{B}{M})x(0) + \dot{x}(0)}{(s^2 + \frac{B}{M}s + \frac{K}{M})}.$$

Example 1 - CASE 1

CASE 1: Initial Conditions are zero, and $f_a(t)=10$

Initial conditions are zero, and hence, we need to determine the forced response due to the forcing function only.

$$f_a(t) = 10 \Rightarrow F_a(s) = \frac{10}{s}$$

$$X(s) = \frac{(\frac{1}{M})F_a(s)}{(s^2 + \frac{B}{M}s + \frac{K}{M})} = \frac{(\frac{10}{M})}{s(s^2 + \frac{B}{M}s + \frac{K}{M})} = \frac{N(s)}{D(s)}$$

Example 1 - Function for ODE Solver

```
function dzdt = MechEx1(t,z)

M=1;
K=100;
B=2;

fa=10;
%fa=50*sin(2*t);

dzdt = [ z(2); -(B/M)*z(2)-(K/M)*z(1)+(1/M)*fa ];
```

Example 1

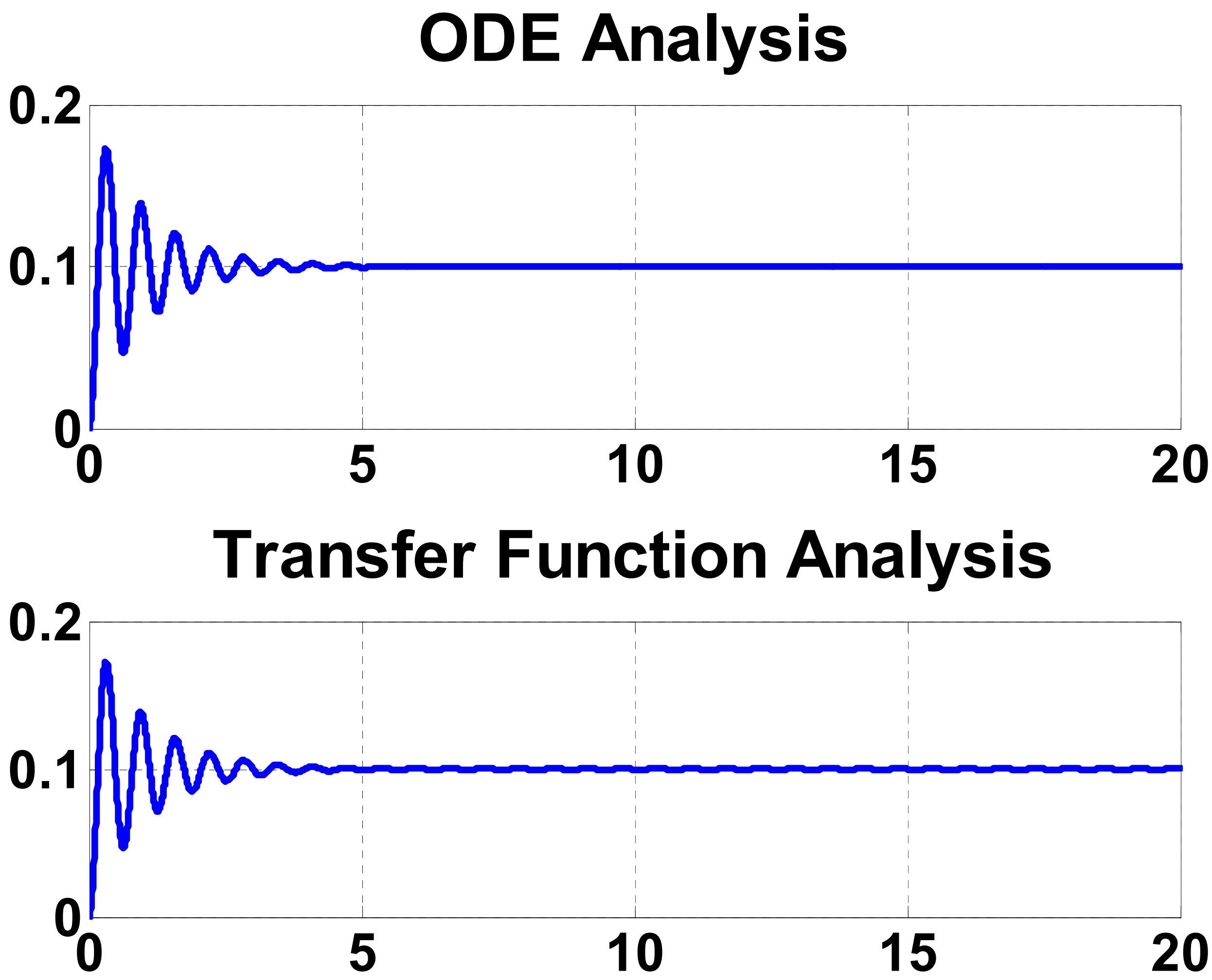
CASE 1 – MATLAB Code

```
clear all, clc

t=0:0.005:20; % Simulation Time

%ODE analysis
[t1,z_ode] = ode45('MechEx1',t,[0;0]);

% transfer function Analysis
M=1; K=100; B=2;
fa=10;
num = [fa/M]; den = [1 B/M K/M 0];
sys=tf(num,den);
[x_tf, t2]=impulse(sys, t);

subplot(2,1,1), plot(t1,z_ode(:,1)),title('ODE Analysis')
subplot(2,1,2), plot(t2,x_tf),
title('Transfer Function Analysis')
```

$$X(s) = \frac{(\frac{10}{M})}{s(s^2 + \frac{B}{M}s + \frac{K}{M})}$$

Example 1

CASE 1 – MATLAB Simulation Results



Example 1 - CASE 2

CASE 2: Initial Conditions are zero, and $f_a(t)=50\sin(2t)$

Some sinusoidal force is applied, and initial conditions are zero, therefore, in order to perform s-domain analysis, we proceed as follows:

- Generate the input signal using `gensig`
- Determine the **transfer function** for the system
- Simulate the system using `lsim`

$$H(s)=\left.\frac{X(s)}{F_a(s)}\right|_{ICs=0}=\frac{N(s)}{D(s)}=\frac{\left(\frac{1}{M}\right)}{\left(s^2+\frac{B}{M}s+\frac{K}{M}\right)}$$

Example 1

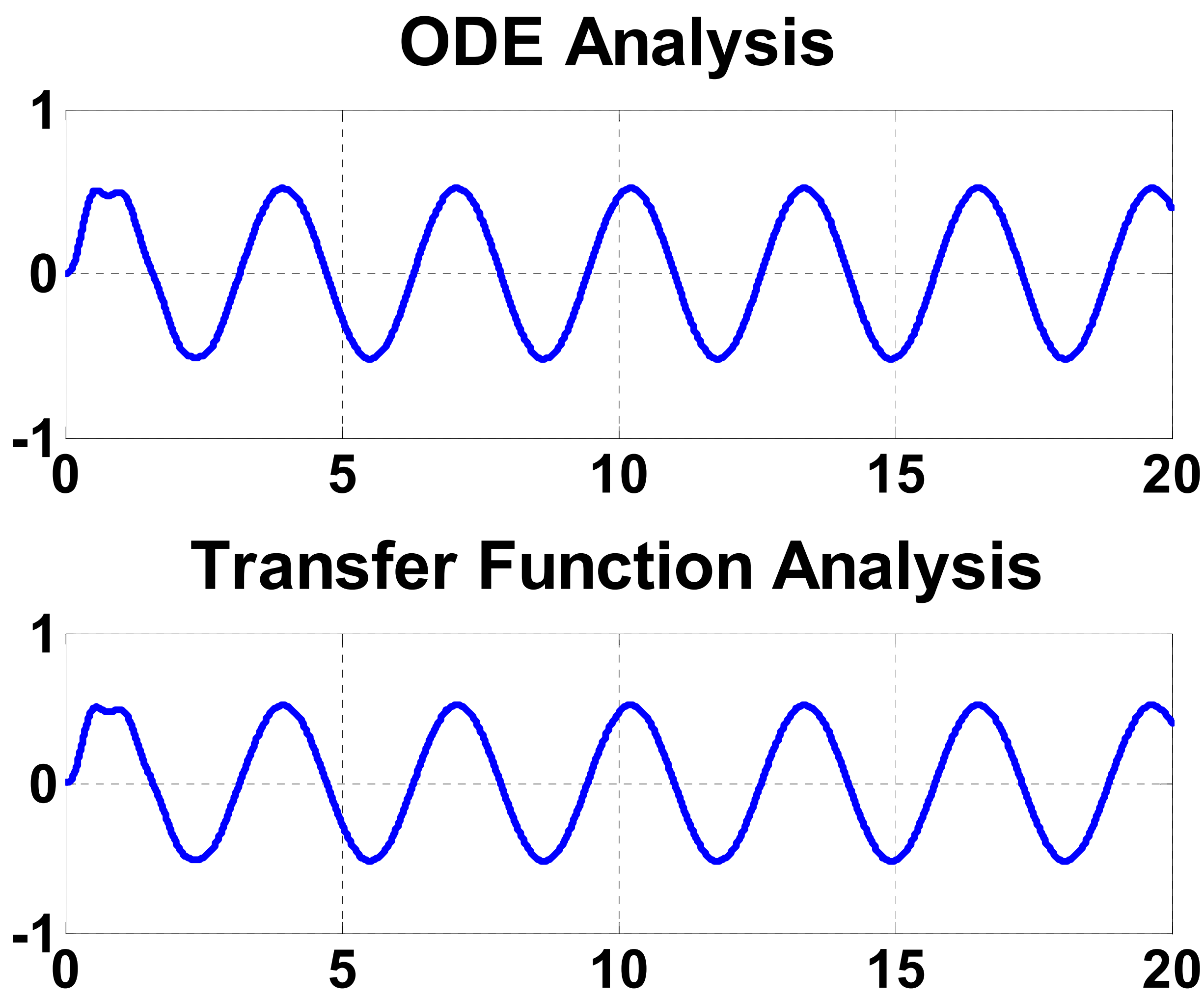
CASE 2 – MATLAB Code

```
clear all, clc

t=0:0.005:20; % Simulation Time
%ODE analysis
[t1,z_ode] = ode45('MechEx1',t,[0;0]);

% transfer function Analysis
% Parameters
M=1; K=100; B=2;
% Generate the input signal fa(t)
[fa,t] = gensig('sine',pi,20,0.005); fa=50*fa;
% System Transfer Function H(s)
num = [1/M];  den = [1 B/M K/M];
sys=tf(num,den);
% System Response
x=lsim(sys,fa,t);
subplot(2,1,1), plot(t1,z_ode(:,1)), title('ODE Analysis')
subplot(2,1,2), plot(t,x), title('Transfer Function Analysis')
```

$$X(s) = \frac{(\frac{1}{M})}{s(s^2 + \frac{B}{M}s + \frac{K}{M})}$$

Example 1

CASE 2 – MATLAB Simulation Results



Example 1 - CASE 3

CASE 3: Non-Zero Initial Conditions and $f_a(t)$=50 sin(2t)

Some sinusoidal force is applied, and initial conditions are non-zero, therefore, in order to perform s-domain analysis, we proceed as follows:

- **Generate the input signal using `gensig`**
- **Determine the transfer function for the system**
- **Trnasfrom the system fuction to state-space form**
- **Simulate the system using `lsim`, with give initial conditions**

$$H(s) = \left.\frac{X(s)}{F_a(s)}\right|_{ICs=0} = \frac{N(s)}{D(s)} = \frac{(\frac{1}{M})}{(s^2 + \frac{B}{M}s + \frac{K}{M})}$$

[A,B,C,D] = tf2ss(num,den)

Example 1

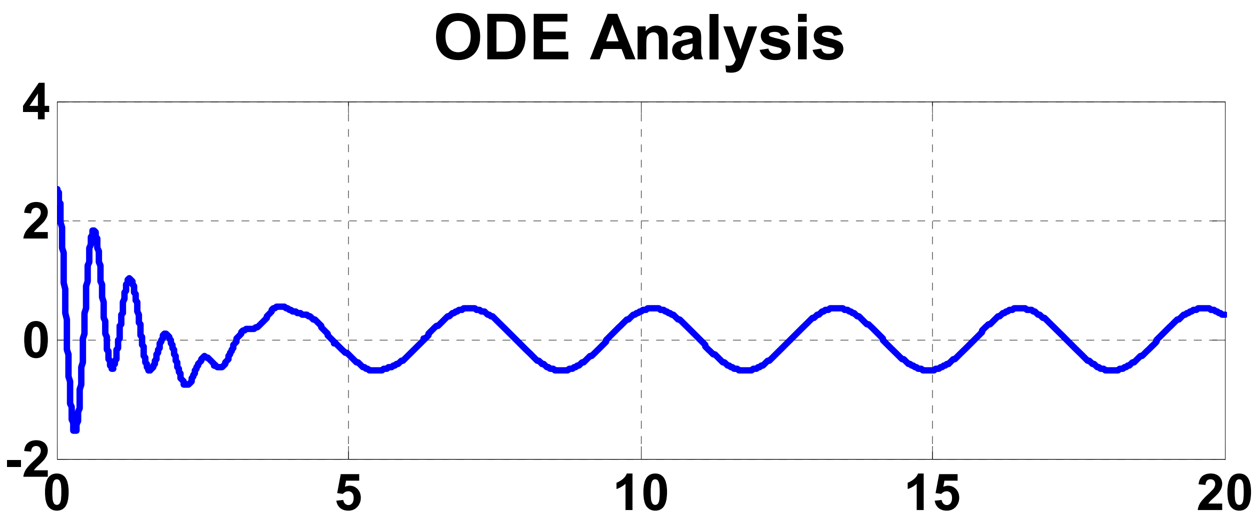
CASE 3 – MATLAB Code

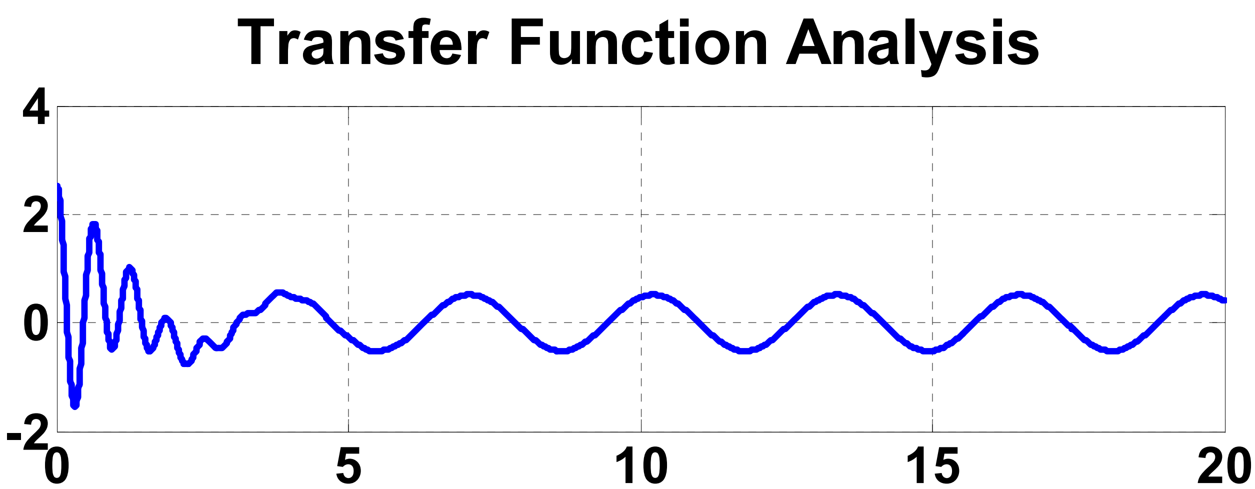
```
clear all, clc
t=0:0.005:20; % Simulation Time
%ODE analysis
[t1,z_ode] = ode45('MechEx1',t,[2.5;2.5]);

% transfer function Analysis
% Parameters
M=1; K=100; B=2;
% Generate the input signal fa(t)
[fa,t] = gensig('sine',pi,20,0.005); fa=50*fa;
% state space system
[A,B,C,D]=tf2ss(num,den)

x=lsim(A,B,C,D,fa,t,[2.5;2.5]);
subplot(2,1,1), plot(t1,z_ode(:,1)), title('ODE Analysis')
subplot(2,1,2), plot(t,x), title('Transfer Function Analysis')
```

$$H(s) = \left.\frac{X(s)}{F_a(s)}\right|_{ICs=0} = \frac{N(s)}{D(s)} = \frac{\left(\frac{1}{M}\right)}{\left(s^2 + \frac{B}{M}s + \frac{K}{M}\right)}$$

Example 1

CASE 3 – MATLAB Simulation Results





Applications of Laplace Transform

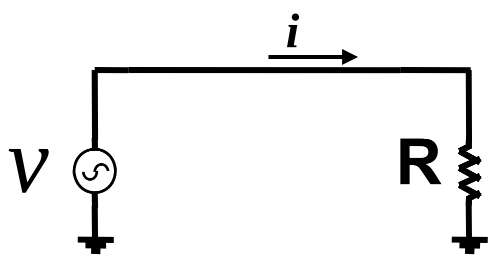
3. Electrical Systems

Variables:

- Voltage v (volts)
- Current i (amperes)

Element Laws:

- **Resistor** (Ohm's law):



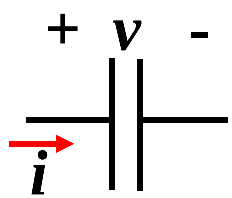
$$v = Ri$$

R resistance in ohms

- Open Circuit ➔ R = infinite
- Short circuit ➔ R = zero

Element Laws:

- **Capacitor**



$$q = Cv$$

C capacitance in farad

Curren through capacitor is given as

$$i = \frac{dq}{dt} = \frac{d(Cv)}{dt} = C\frac{dv}{dt}$$

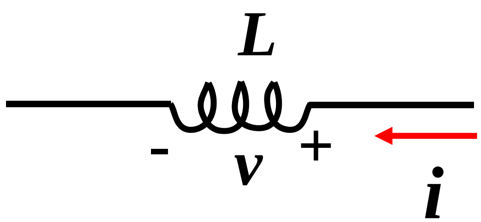
$$\Rightarrow dv = \frac{1}{C} i dt,$$

by integrating, the voltage across capcitor is given as

$$v(t) = v(t_0) + \frac{1}{C}\int_{t0}^{t} i(\lambda)d\lambda$$

Element Laws:

- **Inductor**



L **inductance in henrys**

$$v = L\frac{di}{dt}$$

which can be re - arrnaged as

$$di = \frac{1}{L} v dt$$

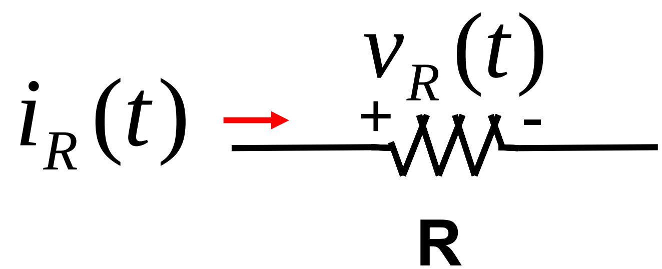
which can be solved as

$$i(t) = i(t_0) + \frac{1}{L}\int_{t0}^{t} v(\lambda) d\lambda$$

s-Domain Circuit Analysis

- **RESISTOR**

$$v_R(t) = R\, i_R(t)$$



Transforming this equation, we get

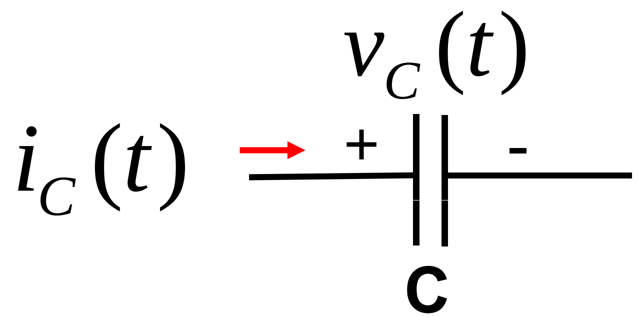
$$V_R(s) = R\, I_R(s)$$

$$or \qquad I_R(s) = \tfrac{1}{R} V_R(s)$$

s-Domain Circuit Analysis

- **CAPACITOR**

$$v_C(t) = v_C(t_0) + \frac{1}{C}\int_{t0}^{t} i_C(\lambda)\,d\lambda$$



or $$i_C(t) = C\,\frac{dv_C(t)}{dt}$$

Transforming this equation, we get

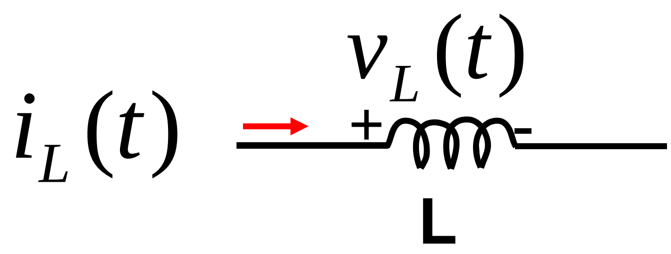
$$V_C(s) = \frac{v_c(t_o)}{s} + \frac{1}{sC} I_C(s)$$

or $$I_C(s) = CsV_C(s) - Cv_C(t_0)$$

s-Domain Circuit Analysis

- **INDUCTOR**

$$v_L(t) = L\frac{di_L}{dt}$$



$$\text{or} \qquad i_L(t) = i_L(t_0) + \frac{1}{L}\int_{t_0}^{t} v_L(\lambda)\,d\lambda$$

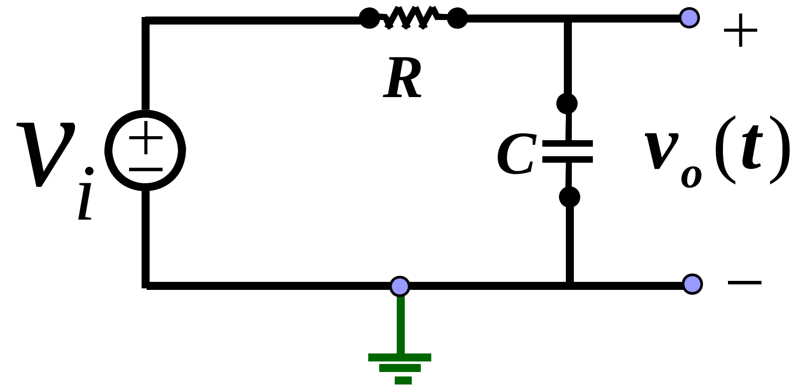
Transforming this equation, we get

$$V_L(s) = sL\,I_L(s) - Li(t_0)$$

$$\text{or} \qquad I_L(s) = \frac{i_L(t_0)}{s} + \frac{V_L(s)}{sL}$$

Example 1

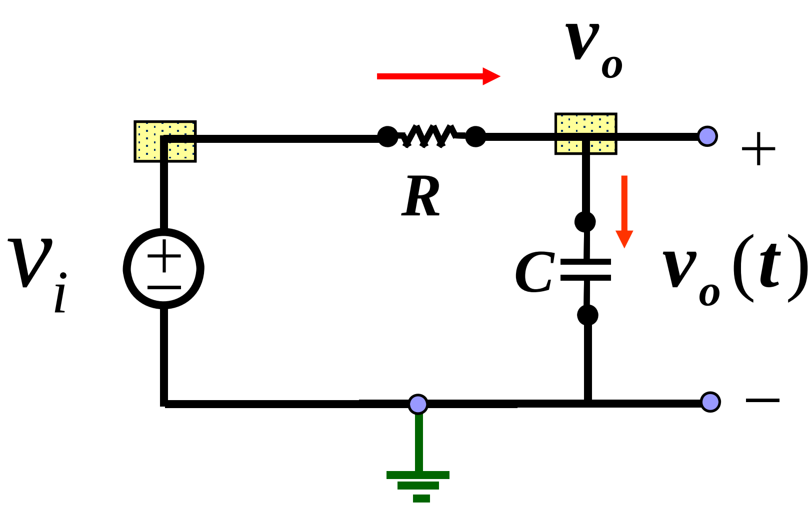
For the circuit shown in the Figure,
v_i = 10V,
R = 10,000 Ω,
C = 10μF.



Using Nodal analysis, find the mathematical model that represents the circuit considering the initial conditions equal to zero and a time interval of 0 to 20 ms. Use Matlab to find the numerical solution of the model.

Solution:

Using Nodal Analysis:

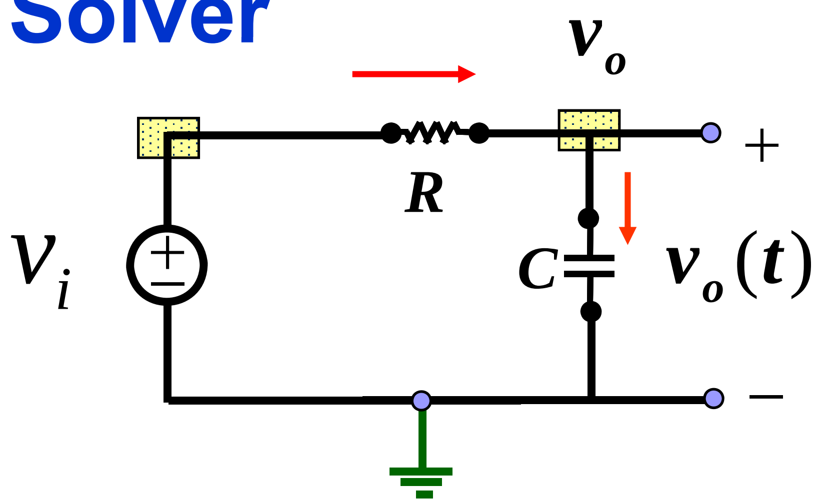


$$\frac{1}{R}(v_i - v_o) = C\frac{dv_o(t)}{dt}$$

$$\frac{dv_o(t)}{dt} = \frac{v_i}{CR} - \frac{v_o(t)}{CR}$$

Simulation Using ODE Solver

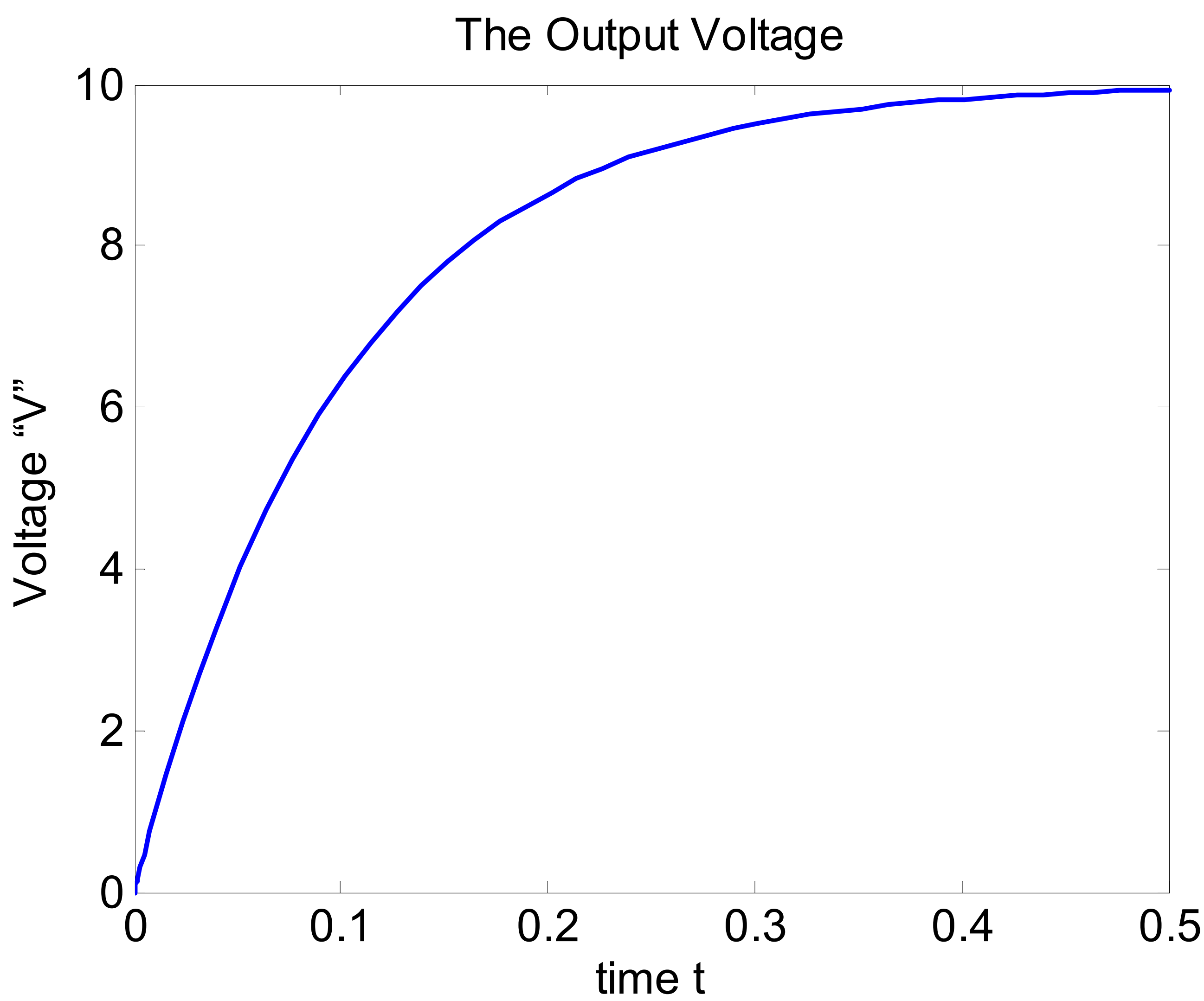
$$\frac{dv_o(t)}{dt} = \frac{v_i}{CR} - \frac{v_o(t)}{CR}$$




```
clear
[t,v] = ode45('ELEC_Ex1',[0 0.5],[0]);
plot(t,v(:,1))
title('The Output Voltage');
xlabel('time t');
ylabel('Voltage "V"');
```

```
function dvdt = ELEC_EX1(t,v)
Vi =10;
R = 10000;
C = 10e-6;
dvdt = [( Vs / (C*R) ) -  ( v(1) / (C*R))];
```

Simulation Results Using ODE Solver



Example 1 – Analysis in s-Domain

$$V_o(s) = \frac{\left(\frac{1}{RC}\right)\left[V_i(s) + v_C(t_0)\right]}{\left(s + \frac{1}{RC}\right)}$$

Voltage Transfer Function

$$H_v(s) = \frac{V_o(s)}{V_i(s)} = \frac{\left(\frac{1}{RC}\right)}{\left(s + \frac{1}{RC}\right)}$$

Example 1 – Analysis in s-Domain

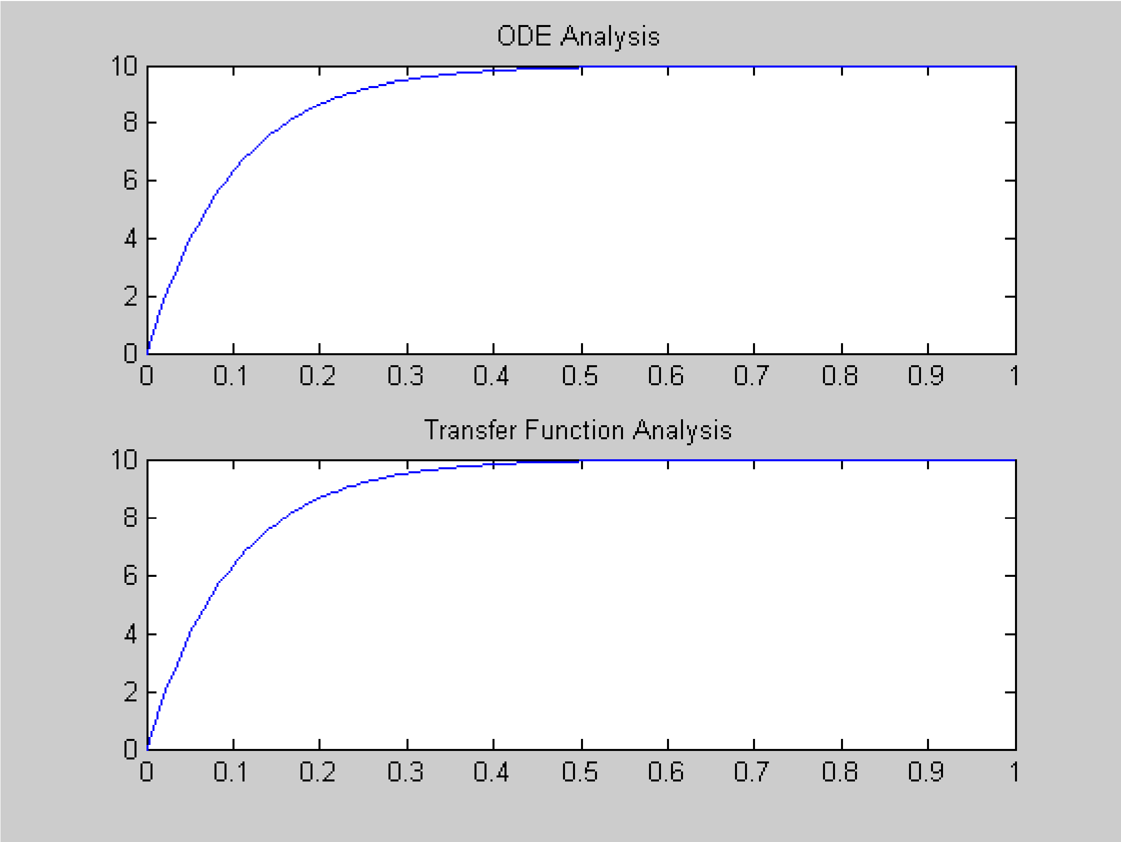
CASE 1:
v_i = 10V,
R = 10,000 Ω,
C = 10μF
Vc(t0) = 0
Capacitor is initially discharged

```
clear all, clc
t=0:0.005:1; % Simulation Time
% ODE analysis
[t1,v] = ode45('ELEC_Ex1',t,[0]);

% transfer function Analysis
R=10e3;  C=10e-6;


num2 = [10/(R*C)]; den2 = [1 1/(R*C) 0];
sys2=tf(num2,den2);

[vc, t3]=impulse(sys2, t);
subplot(3,1,1), plot(t1,v(:,1)),
title('Transfer Function Analysis')
subplot(3,1,2), plot(t3,vc),
```

$$V_o(s) = \frac{\left(\frac{1}{RC}\right)10}{s\left(s+\frac{1}{RC}\right)}$$

Example 1 – Analysis in s-Domain

CASE 1:
v_i = 10V,
R = 10,000 Ω,
C = 10μF
Vc(t0) = 0
Capacitor is initially discharged



$$V_o(s) = \frac{\left(\frac{1}{RC}\right)10}{s\left(s+\frac{1}{RC}\right)}$$

Example 1 – Analysis in s-Domain

CASE 2:

v_i = 10 sin(2*pi*50*t),

R = 10,000 Ω,

C = 10μF

Vc(t0) = 0

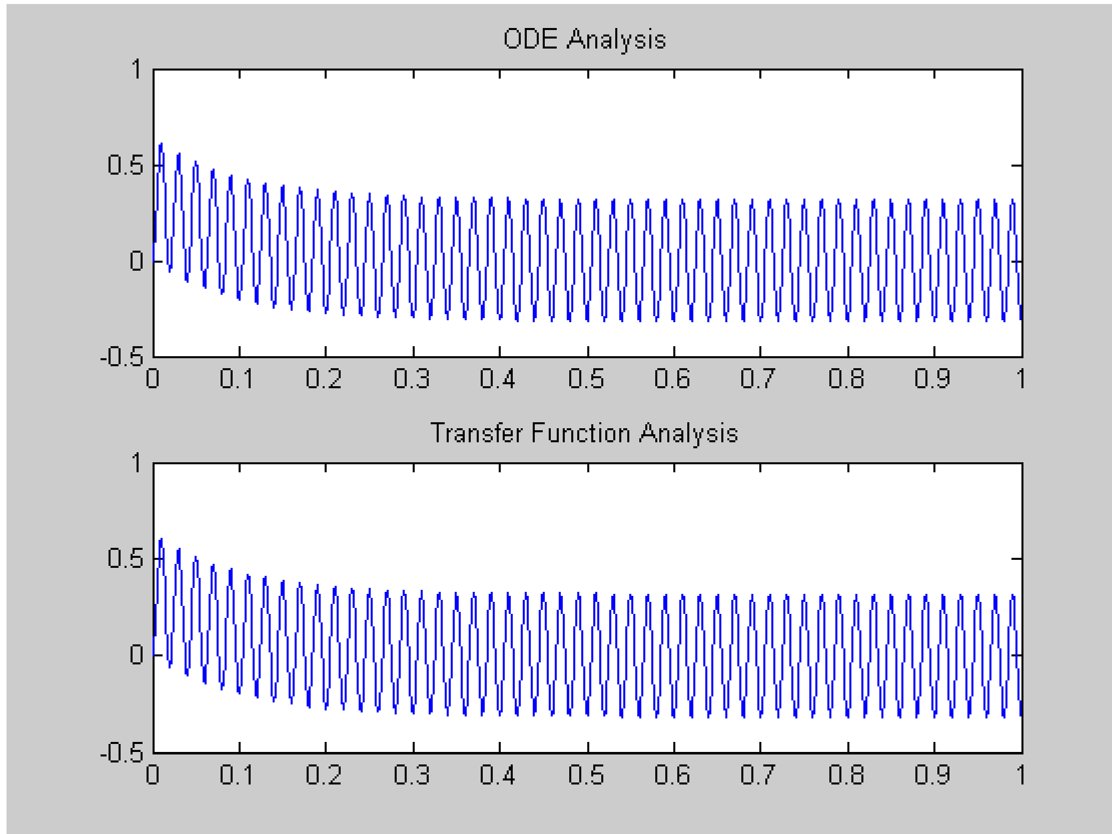
Capacitor is initially discharged

```
clear all, clc
t=0:0.001:1; % Simulation Time
% ODE analysis
[t1,v] = ode45('ELEC_Ex1',t,[0]);
% transfer function Analysis
% Generate the input signal
tau=1/50;
[vi,t2] = gensig('sine',tau,1,0.001);
vi=10*vi;
% Parameters
R=10e3;  C=10e-6;
num2 = [1/(R*C)]; den2 = [1 1/(R*C)];
sys2=tf(num2,den2);

vc=lsim(sys2,vi,t2);
```

$$H_v(s) = \frac{\left(\frac{1}{RC}\right)}{\left(s + \frac{1}{RC}\right)}$$

Example 1 – Analysis in s-Domain

CASE 2:
v_i = 10 sin(2*pi*50*t),
R = 10,000 Ω,
C = 10μF
Vc(t0) = 0
Capacitor is initially discharged



$$H_v(s) = \frac{\left(\frac{1}{RC}\right)}{\left(s + \frac{1}{RC}\right)}$$