These are extra note I wrote for the course of "Introduction to Particle Physics", that I taught during the spring of 2012.

[89 Pages]

Lecture notes On Introduction to Particle Physics.

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Introduction I

Our knowledge of the Physics at distances shorter than $fm := 10^{-15}\ m$, i.e. the sub-nuclear scale is derived from the study of the outcome of the collision of elementary particles at high energies.

- The key parameters for a HEP collider are:

  (1) Energy of the beam
  (2) Intensity of the beam, called Luminosity (more later).

- There are two commonly used reference frames:

  (a) The center of mass frame (CM): $\vec{p}_1 + \vec{p}_2 = 0$

$$s = \left(E_1^{(cm)} + E_2^{(cm)}\right)^2 \tag{1}$$

  (b) The laboratory frame (Lab): $\vec{p}_2 = 0$

Introduction II

$$s = m_1^2 + m_2^2 + 2m_2 E_1^{(Lab)} \xrightarrow{E_1^{(Lab)} >> m_1, m_2} 2m_2 E_1^{(Lab)} \tag{2}$$

Thus, At high energies (much larger than the mass of the beam particles)

$$\begin{aligned} \sqrt{s}(\text{collider}) &= (E_1^{(cm)} + E_2^{(cm)}) \\ \sqrt{s}(\text{fixed target}) &\simeq \left(\frac{2mE}{GeV^2}\right) \text{ GeV} \end{aligned} \tag{3}$$

For instance, colliding two proton beams with energies $E_1 = E_2 = TeV$, yields

$$\sqrt{s_{pp}}(\text{collider}) = 2 \text{ TeV} \tag{4}$$

whereas, a colliding a beam of proton with energy $E_{\text{beam}} = 1\ TeV$ on a fixed target (such a carbon foil), yields

$$\sqrt{s_{pp}}(\text{fixed target}) = \sqrt{200} \text{ GeV} \simeq 44.7 \text{ GeV} \tag{5}$$

Introduction III

- Some kinematics
  Let us consider the process

$$X \to 1 + 2 \tag{6}$$

where X can be a single particle for the case of a decay or two particles colliding in the case of scattering process. In the zero momentum of X (which is the centre of mass system if X represents two colliding particles) we have

$$\vec{p}_1 = -\vec{p}_2 := \vec{k}_f \tag{7}$$

Energy momentum conservation implies that $P_X = P_1 + P_2$, which implies that

$$(P_X - P_1)^2 = P_1^2 \Rightarrow E_{cm}^2 + m_1^2 - 2E_{cm}E_1 = m_2^2 \tag{8}$$

Hence,

Introduction IV

$$E_1 = \frac{s + m_1^2 - m_2^2}{2\sqrt{s}} \tag{9}$$

Similarly, for particle 2 (just exchange m_1 with m_2) we get

$$E_2 = \frac{s + m_2^2 - m_1^2}{2\sqrt{s}} \tag{10}$$

Using $E_1^2 = k_f^2 + m_1^2$, we can solve for k_f and we obtain

$$k_f = \frac{\sqrt{(s - m_1^2 - m_2^2) - 4m_1^2 m_2^2}}{2\sqrt{s}} \tag{11}$$

It is convenient to introduce the function

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$$\lambda(x^2, y^2, z^2) = (x^2 - y^2 - z^2) - 4y^2z^2 \tag{12}$$

known as the **Kallen** function, and write

$$k_f = \frac{1}{2\sqrt{s}}\sqrt{\lambda(s, m_1^2, m_2^2)} \tag{13}$$

- **Mandelstam variables**
  For the scattering of two particles, i.e. the process

$$1 + 2 \to 3 + 4 \tag{14}$$

There are three independent Lorentz-invariant kinematic variables, called the **Mandelstam variables**:

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$$
\begin{aligned}
s &= (p_1 + p_2)^2 = (p_3 1 p_4)^2 && (15) \\
t &= (p_1 - p_3)^2 = (p_2 - p_4)^2 \\
u &= (p_1 - p_4)^2 = (p_2 - p_3)^2 && (16)
\end{aligned}
$$

Their sum is

$$
\begin{aligned}
s + t + u &= 3m_1^2 + m_2^2 + m_3^2 + m_4^2 + 2p_1 p_2 - 2p_1(p_3 + p_4) && (17) \\
&= 3m_1^2 + m_2^2 + m_3^2 + m_4^2 + 2p_1 p_2 - 2p_1(p_1 + p_2) && (18) \\
&= m_1^2 + m_2^2 + m_3^2 + m_4^2 && (19)
\end{aligned}
$$

Introduction VII

- **Mandelstam variables in the high energy limit** Here one can neglect the masses of the particles (i.e. $m_1 = m_2 = m_3 = m_4 = 0$), and in this case we have

$$\begin{aligned} |\vec{p}_{1,2}| &= \frac{\sqrt{E_{cm}}}{2}(1; 0, 0, \pm 1) \\ |\vec{p}_{3,4}| &= \frac{\sqrt{E_{cm}}}{2}(1; \pm \sin\theta, 0, \pm\cos\theta) \end{aligned} \tag{20}$$

where θ is the angle between the momentum of the incident particle 1 and the scattered particle 3. The Mandelstam variables read

$$\begin{aligned} s &= E_{cm}^2 \\ t &= -2p_1p_3 = -\frac{s}{2}(1-\cos\theta) \\ u &= -2p_2p_4 = -\frac{s}{2}(1+\cos\theta) \end{aligned} \tag{21}$$

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- **Mandelstam variables in the massive case** Let us define

$$\beta_{ij} = \frac{\sqrt{\lambda(s, m_3^2, m_4^2)}}{s} \tag{22}$$

Then, we have

$$\begin{aligned} p_{3,4} &= \frac{\sqrt{s}}{2}\left(1 \pm \frac{m_3^2 - m_4^2}{s}; \pm\beta_{34}\sin\theta, 0, \pm\beta_{34}\cos\theta\right) \quad (23) \\ t &= m_1^2 + m_3^2 - \frac{s}{2}\left(1 + \frac{m_1^2 - m_2^2}{s}\right)\left(1 + \frac{m_3^2 - m_4^2}{s}\right) + \frac{s}{2}\beta_{12}\beta_{34}\cos\theta \end{aligned}$$

Introduction IX

- Challenges for colliding beams:

  (a) The particle density in a beam is much lower than in a fixed target (solid or liquid). Thus, one tries the means many times and maximizes the beam intensities (i.e. the number of particle bunches per beam).

  (b) In order to avoid beam-gas interaction (unintended fixed target collisions), a high vacuum is needed in the beam pipe (very low pressure 10^{-9} Pa, which is about 10^{-14} pressure at the sea level.

Introduction X

- Examples of HEP experiments

Experiment (start)	Location	Beams	Beams energies
CESR (1979-)	Cornell Univ	$e^+ \oplus e^-$	$\sqrt{s^{(max)}} = 12$ GeV
LEP (1989-200)	CERN	$e^+ \oplus e^-$	$\sqrt{s} =$ 90 - 200 GeV
SLC (1989-1998)	SLAC	$e^+ \oplus e^-$	$\sqrt{s} \simeq$ 89 - 93 GeV
HERA (1990-2007)	DESY	$e^{\pm} \oplus$ p	$27.5 + 920$ GeV
Tevatron (1983-2011)	Fermilab	$p \oplus \bar{p}$	$\sqrt{s} \simeq 2\ TeV$
LHC (2008- present)	CERN	$p \oplus p$	$\sqrt{s} \simeq 7 - 14\ TeV$

(At HERA, $\sqrt{s}(e^{\pm} \oplus p) = 320\ eV$.)

Transition rate I

- Let $|i>$ and $|f>$ denote the initial state of a particle or system of particles and the final multiple particles final state, respectively. We chose the (relativistic) normalization condition of a single particle state as

$$< \vec{p}|\vec{p}' >= (2\pi)^3 2p^0 \delta^{(3)}(\vec{p} - \vec{p}') \tag{24}$$

The dynamics of the transition from an initial state $|i>$ to some final state $|f>$ is represented by the operator $\hat{\mathcal{S}}$, defined as

$$< f|\hat{\mathcal{S}}|i >\equiv \mathcal{S}_{fi} = \delta_{fi} + i(2\pi)^4 \delta^{(4)}(p'_f - p_i).\mathcal{T}_{fi} \tag{25}$$

where p_i and p'_f are the 4-momenta of the initial and the final particle states, $\mathcal{T}_{fi}$ represents the scattering amplitude of the process, which can be calculated using QFT techniques, namely, Feynman diagrams. Note that the term $\delta^{(4)}(p_f - p_i)$ insures that the energy and momentum are conserved in the transition[1]. Thus, the probability for such process to occur, is given by

Transition rate II

$$\mathcal{P}(|i> \to |f>) = \frac{|\mathcal{S}_{fi}|^2}{<f|f><i|i>}, \quad |i> \neq |f> \tag{26}$$

- To calculate the transition probability for the process, we need to evaluate the square of the δ-distribution. A convenient way to do that is to put the system in a box of size.Now we can write the square of the delta distributions as

$$\begin{aligned} \left[(2\pi)\delta(p_f^0 - p_i^0)\right]^2 &= \left[\int_T t dt \; e^{i(p_f^0 - p_i^{(0)})t}\right] 2\pi\delta(p_f^0 - p_i^0) \\ &= T.2\pi\delta(p_f^0 - p_i^0) \end{aligned}$$

and

$$\begin{aligned} \left[(2\pi)^3\delta^{(3)}(\vec{p}_f - \vec{p}_i)\right]^2 &= \left[\int_V d^3x \; e^{i(\vec{p}_f - \vec{p}_i)t}\right] (2\pi)^3\delta^{(3)}(\vec{p}_f - \vec{p}_i) \\ &= V.(2\pi)^3\delta(\vec{p}_f - \vec{p}_i) \end{aligned}$$

Transition rate III

Considering the fact that in a box of volume $V = L^3$, the states are normalised as

$$< j|j >= (2E_j V)^{1/2} \tag{27}$$

we have

$$\mathcal{P}(|i > \rightarrow |f >) = (V.T)(2\pi)^4 \delta^{(4)}(p_f - p_i). \prod_{j=1}^{N_i} \frac{1}{2E_j V}\Big) \prod_{f=1}^{N_f} \left(\frac{1}{2E_f' V}\right) |\mathcal{T}_{fi}|^2 \tag{28}$$

where N_i and N_f are the number of particles in the initial and final state, respectively. Thus, the transition probability per unit time, i.e. $\mathcal{P}(|i > \rightarrow |f >)/T$, reads

Transition rate IV

$$\omega_{fi} = V(2\pi)^4\delta^{(4)}(p_f - p_i).\prod_{j=1}^{N_i}\left(\frac{1}{2E_jV}\right)\prod_{f=1}^{N_f}\left(\frac{1}{2E'_fV}\right)|\mathcal{T}_{fi}|^2 \quad (29)$$

The above expression assumes that the final state has a well defined quantum numbers. However, since the angle and the momentum of the final state particles are only known up to some accuracy, we define

$$d\omega_{fi} = V.(2\pi)^4\delta^{(4)}(p_f - p_i).\prod_{j=1}^{N_i}\left(\frac{1}{2E_jV}\right)\prod_{f=1}^{N_f}\left(\frac{1}{2E'_fV}\right)|\mathcal{T}_{fi}|^2.d\Phi_f$$

with $d\Phi_f$ the number of states with momentum $\vec{p}_f$ and $\vec{p}_f + d^3\vec{p}_f$, with $f = 1, ..., N_f$, and so it is an element of volume in the phase space. To compute $d\Phi_f$, we recall that the momenta are multiples of $2\pi/L$ (i.e. quantized) in each direction[2]:

Transition rate V

$$\vec{p} = \frac{2\pi}{L}(n_1, n_2, n_3), \quad n_{1,2,3} \in \mathbb{Z} \tag{30}$$

which implies that

$$d^3 n = \left(\frac{L}{2\pi}\right)^3 d^3\vec{p} \equiv \frac{V}{(2\pi)^3} d^3\vec{p} \tag{31}$$

Hence, the transition probability for the final state particles to have momenta between $\vec{p}_f'$ and $\vec{p}_f' + d^3\vec{p}_f'$ is

$$d\omega_{fi} = \left[V \prod_{j=1}^{N_i} (2E_j V)^{-1} \right] .|\mathcal{T}_{fi}|^2 . \mathcal{D}_{N_f}$$

Transition rate VI

where

$$\mathcal{D}_{N_f} = (2\pi)^4 \delta^{(4)}(p_f - p_i) \prod_{\text{out}} \frac{d^3\vec{p}_f}{(2\pi)^3 2E_f}$$

is the relativistic density of final states. Note that $|\mathcal{T}_{fi}|^2$ and $\mathcal{D}$ are Lorentz invariant quantities.

(a) $N_f = 2$ **(two particles in the final state)**
The integration over $\vec{p}_2'$ gives

$$\begin{aligned} \mathcal{D}_2 &= \frac{d^3\vec{p}_1'}{(2\pi)^3 4E_1' E_2'} \delta(E_1' + E_2' - E_{in}) \\ &= \frac{p_1'^2 dp_1' d\Omega}{(2\pi)^3 4E_1' E_2'} \delta(E_1' + E_2' - E_{in}) \end{aligned} \quad (32)$$

Transition rate VII

where E_{in} is the total energy of the initial state particles. In the zero momentum frame, i.e. $\vec{P}_{in} = \vec{p}_1' + \vec{p}_2' = \vec{P}_{out} \equiv 0$, we can write

$$\delta(E_1' + E_2' - E_{in}) = \frac{1}{|\frac{\partial E_1'}{\partial p'} + \frac{\partial E_2'}{\partial p'}|_{p'=k'}} \delta(p' - k') \tag{33}$$

with $p' = |\vec{p}_1'| = |\vec{p}_2'|$, and k' is the value of the momentum for which $[E_1'(k') + E_2'(k') - E_{in}] = 0$. Using the fact that

$$\frac{\partial E_1'}{\partial p'}|_{k'} = \frac{k}{E_1'(k')}, \quad \frac{\partial E_2'}{\partial p'}|_{k'} = \frac{k}{E_2'(k')} \tag{34}$$

then the integration over dp' yields

$$\mathcal{D}_2 = \frac{d\Omega}{16\pi^2} \frac{k'}{E_{\rm in}} \tag{35}$$

Transition rate VIII

(b) $N_f = 3$ (Three particles in the final state)
In this case we have 9 variables (three 3-momenta of the final state particles) subject to 4 constraints (energy momentum conservation). in the rest frame of the three particles we have $\vec{p}_1' + \vec{p}_2' + \vec{p}_3' = 0$, and so one of the 3-momentum, say $\vec{p}_3'$, is determined in terms of ($\vec{p}_1'$ and $\vec{p}_2'$. Moreover, the momenta $\vec{p}_1'$ and $\vec{p}_2'$ are related by the energy conservation constraint:

$$E_{\text{in}} = E_1'(p_1') + E_2'(p_2') + E_3'(|\vec{p}_1' + \vec{p}_2'|) \tag{36}$$

where E_{in} is assumed to be specified. After integration over $\vec{p}_3'$, the relativistic density of the final state reads

$$\mathcal{D}_3 = \frac{1}{(2\pi)^5 8 E_1' E_2' E_3'} \, p_1'^2 dp_1' d\Omega_1 p_2'^2 dp_2' d\Omega_{12} \delta(E_1' + E_2' + E_3' - E_{\text{in}}) \tag{37}$$

where Ω_{12} is the solid angle of $\vec{p}_2'$ relative to $\vec{p}_1'$, parametrised by angles θ_{12} and ϕ_{12} such that

Transition rate IX

$$d\Omega_{12} = d\cos\theta_{12} d\phi_{12} \tag{38}$$

We will eliminate $\cos\theta_{12}$ using the energy conservation constraint:

$$\int d\cos\theta_{12}\delta(E'_1 + E'_2 + E'_3 - E_{\text{in}}) \tag{39}$$

Using

$$E'_3 = \sqrt{(\vec{p}'_1 + \vec{p}'_2)^2 + m_3^2} = \sqrt{p_1'^2 + p_2'^2 + 2p'_1 p'_2 \cos\theta_{12} + m_3^2} \tag{40}$$

we obtain that

$$\mathcal{D}_3 = \frac{1}{8(2\pi)^5} \frac{p'_1 dp'_1}{E'_1} \frac{p'_2 dp'_2}{E'_2} d\Omega_1 d\phi_{12} \tag{41}$$

Transition rate X

Using the fact that for relativistic particles $EdE = pdp$, the density of state for a three body final states in the zero-momentum frame reads

$$\mathcal{D}_3 = \frac{d\Omega_1 d\phi_{12}}{256\pi^2} dE_1' dE_2' \quad (42)$$

- **Decay rate**
  The process such as

$$X \rightarrow 1 + 2 + ...N_f$$

is called decay of X into N_f particles. The total decay rate, Γ_X, can be obtained by setting $N_i = 1$ in the above expression of the transition rate and integrating over all possible final state momenta:

Transition rate XI

$$\Gamma_X = \frac{1}{2E_X} \int d\tilde{p}'_1 ... d\tilde{p}'_{N_f} (2\pi)^4 \delta^{(4)}(p_f - p_i) |\mathcal{T}_{fi}|^2 \tag{43}$$

where

$$d\tilde{p}'_f := \frac{d^3 p'_f}{2E'_f (2\pi)^3} \tag{44}$$

Therefore, the life time of the particle X is

$$\tau_X = \frac{1}{\Gamma_X}$$

Note that the term under the integral in Eq (43) is Lorentz invariant where as E_X is not. Hence, τ_X defined above is the life time as measured in the rest frame of the particle X and in another reference frame it will be different[3].

Transition rate XII

For $N_f = 2$, and in the rest frame of the decaying particle (of mass M_X), we have

$$\begin{aligned} \Gamma^{(0)}(X \to 1+2) &= \frac{1}{2M_X} \int \left|\mathcal{T}_{X\to 1+2i}\right|^2 \mathcal{D}_2 \\ &= \frac{1}{32\pi^2 M_X^2} k_{cm} \int d\Omega \left|\mathcal{T}_{X\to 1+2i}\right|^2 \end{aligned} \quad (45)$$

where k_{cm} is the centre of mass momentum of the particles 1 and 2 (i.e. $k_{cm} = k_1^{(cm)} = k_2^{(cm)}$). Hence, the decay rate of X to two final state particles reads

$$\Gamma(X \to 1+2) = \frac{k_{cm}}{8\pi M_X^2} \left|\mathcal{T}_{X\to 1+2i}\right|^2 \quad (46)$$

Transition rate XIII

In the non-zero momentum frame of the particle X the decay rate reads

$$\Gamma_X^{(0)} = \frac{M_X}{E_X}\Gamma_X^{(0)} \equiv \frac{1}{\gamma}\Gamma_X^{(0)} \qquad (47)$$

where $\gamma = 1/\sqrt{1 - v^2/c^2}$ is the usual time-dilatation factor.

Transition rate XIV

- **Cross section**
  Consider the scattering of two particles, say 1 and 2, into a number of a final state particles. We denote $\mathcal{J}_{\rm sc} d\Omega$ the number of outgoing particles per unit time, scattered into an element of solid angle $d\Omega$ in direction (θ, ϕ). This number is proportional to the incident flux Φ_{in}, defined as the number of particles per unit time crossing a unit area normal to the direction of incidence.

  We define the **differential cross section** as

$$\frac{d\sigma}{d\Omega} := \frac{\mathcal{J}_{\rm sc}}{\Phi_{in}} \tag{48}$$

  However,

$$\mathcal{J}_{\rm sc} = \left(\frac{d\omega_{fi}}{d\Omega}\right) \mathcal{N}_{in}$$

Transition rate XV

Here $\mathcal{N}_{in}$ is the number of particles in the incoming beam. Thus, we can write the differential cross section as[4]

$$\frac{d\sigma}{d\Omega} = \left(\frac{d\omega_{fi}}{d\Omega}\right)\left(\frac{1}{\phi_{in}}\right)$$

where

$$\phi_{in} = \frac{\Phi_{in}}{\mathcal{N}_{in}} \Leftarrow \text{Incident flux per unit beam}$$

The cross section can be obtained by integrating over the solid angle

$$\sigma = \int \frac{d\sigma}{d\Omega}\, d\Omega \tag{49}$$

Transition rate XVI

In other words,
the differential cross section is the transition rate for one scattering center divided by the flux of the incident particles on the volume containing this one scattering center.

- To derive the expression of the cross section we will first consider the particle 2 to be at rest. Then, for one particle in some volume V, with a speed $v_1 = \frac{|\vec{p}_1|}{E_1}$, the incident flux per unit beam is given by

$$\phi_{in} = \frac{1}{V}\left|\vec{v}_1\right| = \frac{|\vec{p}_1|}{E_1 V} \tag{50}$$

Transition rate XVII

Thus, It is straightforward to show that $|\vec{p}_1|m_2$ can be expressed in terms of the total center-of-mass energy squared, $s = (E_1 + E_2)^2$ as

$$|\vec{p}_1|m_2 = |\vec{p}_{1\,CM}|\sqrt{s} \tag{51}$$

which implies

$$\frac{d\sigma}{d\Omega} = \frac{1}{4|\vec{p}_1|_{cm}\sqrt{s}}|\mathcal{T}_{fi}|^2\,\mathcal{D}_{N_f} \tag{52}$$

In general, for two colliding particles, with velocities $\vec{v}_1$ and $\vec{v}_2$, respectively, the incident flux per unit beam is given by

$$\phi_{in} = \frac{1}{V}|\vec{v}_1 - \vec{v}_2| = \frac{1}{V}\left|\frac{\vec{p}_1}{E_1} - \frac{\vec{p}_2}{E_2}\right| \tag{53}$$

Transition rate XVIII

Thus, the differential cross section for two colliding particles is

$$d\sigma = \frac{1}{4|E_2\vec{p}_1 - E_1\vec{p}_2|} |\mathcal{T}_{fi}|^2 \, \mathcal{D}_{N_f} \tag{54}$$

In the CM frame of the two colliding particles we have

$$\vec{p}_1 = -\vec{p}_2 = \vec{k}_{\text{in}} \tag{55}$$

Then

$$d\sigma = \frac{1}{4\sqrt{s}k_{\text{in}}} |\mathcal{T}_{fi}|^2 \, \mathcal{D}_{N_f} \tag{56}$$

where $\sqrt{s} =]E_1(k) + E_2(k)]$ is the total energy of the colliding particles in the their centre of mass frame.

The differential cross section (54) can be written in manifestly Lorentz-invariant form as

Transition rate XIX

$$d\sigma = \frac{|\mathcal{T}|^2}{4\sqrt{(p_1 p_2)^2 - m_1^2 m_2^2}} \mathcal{D}_{N_f} \quad (57)$$

The quantity $p_1 p_2)^2 - m_1^2 m_2^2$ can be expressed as

$$\begin{aligned} p_1 p_2)^2 - m_1^2 m_2^2 &= \frac{1}{4}\left(p_1^2 - m_1^2 + p_2^2 - m_2^2 + 2p_1 p_2\right)^2 - m_1^2 m_2^2 \quad (58) \\ &= \frac{1}{4}\left[\left(p_1 + p_2\right)^2 - m_1^2 - m_2^2\right)^2 - 4m_1^2 m_2^2\right]^2 \\ &= \frac{1}{4}\lambda(s, m_1^2, m_2^2) \end{aligned}$$

Thus,

Transition rate XX

$$d\sigma = \frac{|\mathcal{T}|^2}{2\sqrt{\lambda(s, m_1^2, m_2^2)}} \mathcal{D}_{N_f} \qquad (59)$$

- **Two-to-Two scattering**
  Inserting the expression of $\mathcal{D}$ for $N_f = 2$ in Eq (56) we obtain

$$\frac{d\sigma}{d\Omega} = \left[\frac{k_{\text{fin}}}{k_{\text{in}}}\right] \frac{1}{64\pi^2 s} \left|\mathcal{T}_{1+2\to 3+4}\right|^2 \qquad (60)$$

Transition rate XXI

where k_{fin} is the final CM momentum. If the scattering amplitude has no ϕ-dependence, we can express the differential cross section as (the velocities β_{ij} were introduced in Eq(22) in the introduction):

$$d\sigma = \frac{\beta_{34}}{\beta_{12}} \frac{d\cos\theta}{32\pi} |\mathcal{T}|^2 \tag{61}$$

The differential cross section with respect to the Mandelstam variable t can be obtained from the above expression using the chain rule

$$\frac{d\sigma}{dt} = \frac{d\sigma}{d\cos\theta} \frac{d\cos\theta}{dt} \tag{62}$$

Using the expression of the parameter t given in Eq (23), we get

Transition rate XXII

$$\frac{d\sigma}{dt} = \frac{1}{16\pi\beta_{12}^2}\frac{|\mathcal{T}|^2}{s^2} \tag{63}$$

- **There are four type of cross sections:**

  (a) **Elastic** cross section:
  The initial and final state particles are the same. Thus, the kinetic energy and the three-momentum are conserved. In this case the expression reduces to

$$\frac{d\sigma^{(\text{elastic})}}{d\Omega} = \frac{1}{64\pi^2 s}\left|\mathcal{T}_{1+2\to3+4}\right|^2 \tag{64}$$

Transition rate XXIII

(b) **Inelastic** cross section:
The initial and final state particles are different. In this case the kinetic energy and the three-momentum are not conserved. instead, the 4-momentum will be conserved.

(c) **Exclusive** cross section:
It is the cross section of a process with a given final state, and the process will be called exclusive process. For example, at LEP, one search for Higgs boson production along with a Z boson, and so we say that $\sigma(e^+e^- \to ZH)$ is a exclusive cross section. Similarly, at the LHC, the cross section $\sigma(pp \to WH)$ is exclusive. In general, Exclusive cross sections are easy to compute.

(d) **Inclusive** cross section:
Here one sums over all possible exclusive cross sections for a given initial state, e.g. $\sigma(pp \to \text{anything})$. Often, inclusive cross sections are easy to measure, as one does not need to identify which kind of particle has been produced.

Transition rate XXIV

- Units used for cross section in particle Physics:
  As we noted above, the cross section has unit of area, i.e. m^2. However, in particle physics, we use much smaller unit called **barn**, denoted by the letter **b**. By definition,

$$1\ b := 10^{-24}\ cm^2$$

  which a typical size of a heavy nuclei. For instance, the proton-proton cross section is[5]

$$\sigma(pp) \sim 100\ mb \equiv \sigma_{el} + \sigma_{inel}$$

  where $\sigma_{el} \sim 25\ mb$ is the elastic cross section, i.e. $pp \to pp$ (no color flow between the colliding protons), and $\sigma_{inel} \sim 70\ mb$ is the inelastic cross section which results in multi-particle final states (e.g. $pp \to pn^+$, $pp \to (p\pi^+\pi^-)(p\pi^+\pi^-)$,...). Note that, roughly, the value of σ_{pp}

Transition rate XXV

corresponds to the cross section area of a proton[6], πr_p^2, with $r_p \sim$ fm is the classical radius of proton. The cross section for W and Z production at the LHC are of the order **nano barn** (nb).

Transition rate XXVI

Example:

At energies much smaller than the $Z^{(0)}$ gauge boson mass, the cross section for the annihilation of electron-positron to muons- anti-muons is dominated by the EM interaction (QED), and it is given by by

$$\sigma(e^+e^-\mu^+\mu^-) = \frac{87}{s}\ nb :\equiv 1 \text{ R unit} \tag{65}$$

with s is the center of mass energy in GeV^2. Thus, at $\sqrt{s} = 10$ GeV, the cross section for muon production is about $1nb$.

$$\hat{\mathcal{S}} = \mathbb{I} + i\hat{\mathcal{T}}$$

[2]I am taking $\hbar = 1$

[3]In fact, according to the theory of special relativity, it will be longer.

[4]One can also think of the cross section as follows. If the incident beam has flux Φ_{in} (number of particles per unit time per unit area), and the number of particles per unit time and per unit area scattered in the direction (θ, ϕ) is Φ_{sc}, then the total amount of particles scattered through a spherical surface element with area dA at distance D is $\Phi_{sc}\ A$. This amount must be proportional to the incident flux; therefore

$$\Phi_{sc}\ A = \Phi_{in}\ \sigma(\theta, \phi)$$

The quantity $d\sigma(\theta, \phi)$ determines which fraction of the incident flux contributes to the

- If the cross section of a process is σ, and $\mathcal{R}$ is the event rate, i.e. the number of events per seconds, then the luminosity is defined by

$$\mathcal{R} = \mathcal{L}.\sigma \tag{66}$$

which has units of $cm^{-2}.s^{-1}$, and so it is a measure of the number of collisions that can be produced in a detector per cm^2 per second.

- For a fixed target experiment, where the target is homogeneous and larger than the incoming beam, we have

$$\begin{aligned} \mathcal{R} &= n_{\text{target}} V \mathcal{J}_{\text{sc}} \\ &= n_{\text{target}} (A \times l) \Phi_{in} \times \sigma \end{aligned}$$

Here n_{target} is density of scattering center in the target material and l is its length. Writing the incident flux as

$$\Phi_{in} = \frac{\mathcal{J}_{in}}{A}$$

where $\mathcal{J}_{in}$ is the intensity of the incoming beam. Thus, the fixed target luminosity is given by

$$\mathcal{L}_{FT} = \mathcal{J}_{in} n_{\text{target}} l$$

- For a collider, I am not gonna derive the expression of the luminosity it here. I will give you its formula:

$$\mathcal{L}_{\text{Collider}} = \frac{n_B N_L N_R}{A_{\text{eff}}} f_{\text{rev}} \tag{67}$$

where

$$\begin{aligned} n_B &: \quad \text{Number of bunches,} \\ N_L, N_R &: \quad \text{Number of particles per bunch in each direction,} \\ f &: \quad \text{Frequency of revolution around the ring} \\ A_{\text{eff}} &: \quad \text{Effective transverse area of the beam} \end{aligned}$$

For a Gaussian shaped beam with dimensions s_x and s_y, its transverse size is

$$A_{\text{eff}} = 4\pi s_x s_y$$

the accumulated luminosity over a period of time T is

$$L = \int_0^T \mathcal{L} dt \tag{68}$$

which is also called **the integrated luminosity**. From the formula (67), we note that the luminosity can be increased by reducing the cross section of the beam, by increasing the number of particles in the beam or by increasing the revolution frequency.

Example:
The LHC is collides two beams of protons running in a ring of about 27 *km* in circonference, and so

$$f_{\text{rev}} \simeq \frac{3 \times 10^8 \ /s}{27 \times 10^3 \ m} \simeq 10^4 \ HZ$$

It uses $n_B \simeq 2800$, $N_L = N_R \simeq 10^{11}$, and $\sqrt{s_x s_y} = 16$ microns, which yields a luminosity[7]

$$\mathcal{L}^{(LHC)} \sim 10^{34} cm^{-2}.s^{-1} = 10 \text{ nb}^{-1} s^{-1} \qquad (69)$$

Of course it takes time for the machine to reach the peak of the design luminosity. Then, after a year of running with such a design luminosity, the LHC can accumulate

$$L^{(LHC)} \sim 100 \text{ fb}^{-1} \qquad (70)$$

In the table below, we give estimates of the expected number of events at the LHC for different production cross sections at the design luminosity.

Final states	Cross section	# events/s
Total	100 mb	10^9
W or Z	100 nb	10^3
$t\bar{t}$	nb	10
WW or ZZ	0.1 nb	1

- The cross section is determined from experiment via the relation

$$\sigma(\sqrt{s}) := \frac{N_{\text{sel}} - N_{\text{bkg}}}{\epsilon \int \mathcal{L} dt}$$

where N_{sel} and N_{bkg} represent to the number of events passing the selection cuts and the number of background events in the selected sample, and ϵ is the detection efficiency factor which accounts for the trigger efficiency, the geometrical acceptance and the efficiency of the selection cuts.

[7] $1\ barn = 10^{-24}\ m^2$.

Kinematics I

- **Lorentz transformations**:
  Let $P^{\mu}_{\text{Lab}} := (E, \vec{p})$ be the 4-momentum of a particle or system of particles in the laboratory frame. For an observer in frame moving with velocity $\vec{v}$, the viewed 4-momentum will be $(P'^{\mu} := (E', \vec{p}'))$, and is related to P_{Lab} via the Lorentz transformation ($\beta = v/c$):

$$
\begin{aligned}
E' &= \gamma\left(E - \beta p_{\|}\right) \\
p'_{\|} &= \gamma\left(p_{\|} - \beta E\right) \\
p'_{\perp} &= p_{\perp}
\end{aligned}
\qquad (71)
$$

where $\gamma = (1 - \beta^2)^{-1/2}$ is the relativistic factor, and the symbols $\|$ and $\perp$ refer to the components parallel and perpendicular to the velocity $\vec{v}$, respectively.

Kinematics II

- **System of two particles**:

  Consider two particles, say 1 and 2, of masses m_1 and m_2, and have 4-momentum $(E_1, \vec{p}_1)$ and $(E_2, \vec{p}_2)$, respectively, in some reference frame. Then, the center-of-mass energy can be written

$$E_{cm} := \sqrt{(p_1 + p_2)_\mu (p_1 + p_2)^\mu}$$
$$= \sqrt{(E_1 + E_2)^2 - (\vec{p}_1 + \vec{p}_2)^2}$$

  or, equivalently,

$$\boxed{E_{cm} = \sqrt{m_1^2 + m_2^2 + 2E_1 E_2 (1 - \beta_1 \beta_2 \cos\theta_{12})}}$$

  In a frame where particle 2 is at rest, i.e. $\vec{p}_2 = \vec{0}$, we get

Kinematics III

$$E_{cm} = \sqrt{m_1^2 + m_2^2 + 2E_1^{(\mathrm{Lab})} m_2}$$

The velocity of the center of mass of this system of two particles, with the particle 2 being at rest, is given by

$$\vec{\beta}_{cm} = \frac{\vec{p}_1^{(\mathrm{Lab})}}{E_1^{(\mathrm{Lab})} + m_2}, \qquad \gamma_{cm} = \frac{E_1^{(\mathrm{Lab})} + m_2}{E_{cm}} \tag{72}$$

Kinematics IV

- **Rapidity**:
  Let us introduce a new parameter y such that

$$\beta = \tanh y \quad \Rightarrow \quad \gamma = \cosh y, \qquad \beta\gamma = \sinh y$$

This allows us to write the Lorentz transformation as

$$\begin{pmatrix} E' \\ p'_{||} \end{pmatrix} = \begin{pmatrix} \cosh y & -\sinh y \\ -\sinh y & \cosh y \end{pmatrix} \begin{pmatrix} E \\ p_{||} \end{pmatrix}$$

Now, suppose the collision of beam of particles and as a result a particle of mass m is produced with a velocity $\vec{\beta}$ making an angle θ with the beam line. Decomposing its momentum into two components: one along the beam line, $p_{||}$, and the other, p_T, perpendicular to it, the energy of the particle with respect to an observer in the laboratory frame (at rest) can then be expressed as

Kinematics V

$$E = \sqrt{p_{\|}^2 + E_T^2}$$

where

$$E_T = \sqrt{m^2 + p_{\perp}^2}$$ $\Leftarrow$ **Transverse energy**

In a reference frame moving along the beam line with a speed equal to the component of the particle's speed along the beam, i.e. $\beta_{\|} := p_{\|}/E$, the 4-energy momentum seen will be $(m_{\perp}, 0)$. The 4-momentum as seen by an observer at rest in the laboratory, i.e. $(E, p_{\|})$, can be obtained from $(m_{\perp}, 0)$, is given by the Lorentz transformation $(\beta \to -\beta)$ as:

$$\begin{pmatrix} E \\ p_{\|} \end{pmatrix} = \begin{pmatrix} \cosh y & \sinh y \\ \sinh y & \cosh y \end{pmatrix} \begin{pmatrix} m_{\perp} \\ 0 \end{pmatrix} \Rightarrow E = m_{\perp} \cosh y, \quad p_{\|} = m_{\perp} \sinh y$$

Kinematics VI

Hence, we deduce that[8]

$$y = \tanh^{-1}\frac{p_{\parallel}}{E} := \tanh^{-1}\beta_{\parallel} = \ln\frac{E' + p'_{\parallel}}{m_{\perp}}$$

or, equivalently

$$y = \frac{1}{2}\ln\frac{E+p_{\parallel}}{E-p_{\parallel}}$$

This dimensionless variable is called **rapidity**, and as its name suggests, is related to velocity of the particle along the beam line. It can also be re-written in the form

$$y = \frac{1}{2}\ln\frac{1+\beta\cos\theta}{1-\beta\cos\theta}$$

Kinematics VII

where θ is the polar angle between the particle velocity $\vec{\beta}$ and the beam axis, i.e.

$$\tan\theta = \frac{p_\perp}{p_\parallel}$$

- **Element of phase space in term of y and p_T**
  The Lorentz invariant phase space element reads

$$\frac{d^3p}{E} \quad := \quad \frac{dp_\parallel}{E} d^2p_T$$

Now since this quantity is a Lorentz invariant, in particular under a boost along $p_\parallel$, we expect that $dp_\parallel/E$ to be written in terms of the differential element dy. This is indeed the case. We have

Kinematics VIII

$$\begin{aligned} dy &= \left(\frac{\partial y}{\partial p_{\|}} + \frac{\partial y}{\partial E}\frac{\partial E}{\partial p_{\|}}\right) dp_{\|} \\ &= \left(\frac{E}{E^2 - p_{\|}^2} - \frac{p_{\|}}{E^2 - p_{\|}^2}\frac{p_{\|}}{E}\right) dp_{\|} \\ &= \frac{dp_{\|}}{E} \end{aligned}$$

Thus,

$$\frac{d^3p}{E} = d^2\mathbf{p}_T dy$$

Kinematics IX

Assuming an azimuthal symmetry, we can write

$$\frac{d^3p}{E} = \pi dp_T^2 dy$$

One can also define the cross section per Lorentz-invariant phase space, $E\frac{d^3\sigma}{d^3p}$, which can be expressed in terms of the rapidity and transverse momentum as:

$$E\frac{d^3\sigma}{d^3p} = \frac{1}{\pi}\frac{d^2\sigma}{dp_T^2 dy}$$

Kinematics X

Why should we be interested in the rapidity?

To see why, consider a particle of velocity β, and rapidity y with respect to some reference frame, say the lab frame, and another observer moving with a velocity β_* (boosted frame) along the direction of $\vec{\beta}$. Then, the rapidity y' in the boosted frame is given by

$$y' = \ln \frac{E' + p'_{\|}}{m_{\perp}} = \ln \frac{(E \cosh y_* - p_{\|} \sinh y_*) + (-E \sinh y_* + p_{\|} \cosh y_*)}{m_{\perp}}$$

which gives

$$y' = y - y_*$$

Kinematics XI

where y_* is the rapidity of the boosted frame.

This transformation law for the rapidity implies that $\Delta y = \Delta y'$, i.e. the difference of rapidity is a Lorentz invariant. This can also be applied to a system of two particles. To see that suppose that during a collision two particles were ejected, with rapidities y_1 and y_2 measured in some reference frame. In another reference frame moving with velocity β_* parallel to the beam, these particles will have rapidities $y_1' = y_1 - y_*$ and $y_2' = y_2 - y_*$. Hence,

$$y_1' - y_2' = y_1 - y_2 \tag{73}$$

Therefore, the difference between the rapidities of two particles is an invariant under boost transformation along the beam line.

Kinematics XII

It is worth noting that using the transformation law of the rapidity under a boost, and the identity

$$\tanh^{-1} x_1 + \tanh^{-1} x_1 = \tanh^{-1} \frac{x_1 + x_2}{1 + x_1 x_2}$$

we recover the relation for the addition of relativistic velocities, namely

$$\beta' = \frac{\beta - \beta_*}{1 + \beta\beta_*}$$

where β' is the velocity of the particle as seen by an observer moving with a velocity β_* (i.e. in the boosted frame).

Kinematics XIII

- **Rapidity in the center of mass system**

  Consider a a system of two particles, 1 and 2, with equal masses and rapidities y_1 and y_2, respectively, as seen by an observer in some reference frame, say the lab frame. Obviously, in the center-of-mass (CM) frame, the particles will have rapidities equal in magnitude but opposite sign, i,e, $y_1^* = -y_2^*$, where the symbol "*" label the quantities measured by an observer in the CM frame. If y_{CM} is the rapidity of the center of mass frame, then according the transformation law of the rapidities under a boost we have

$$y_1^* = y_1 - y_{CM}, \qquad -y_1^* = y_2 - y_{CM} \tag{74}$$

Thus,

$$y_{CM} = \frac{y_1 + y_2}{2}$$

Kinematics XIV

Also, given The rapidities of the particles 1 and 2 in the center-of-mass frame, can be expressed as

$$y_1^* = -\frac{\Delta y}{2}, \quad y_2^* = \frac{\Delta y}{2}$$

where $\Delta y = (y_2 - y_1)$.

Examples:
(a) In a fixed target experiment:

$$y_{CM} = (y_{\text{target}} + y_{\text{beam}})/2 = y_{\text{beam}}/2$$

(b) At collider (leptons or at hadron level):

$$y_{CM} = (y_{\text{target}} + y_{\text{beam}})/2 = 0$$

Kinematics XV

- **Pseudo rapidity**:
  One problem with using rapidity is that it can be hard to measure for highly relativistic particles. The reason is that component of the momentum along the beam line will be large, and the beam pipe is on the way of measuring it precisely, which makes it very difficult to infer the rapidity. As we will see below, we will define another quantity that is very closely related to rapidity which is much easier to measure.

  For highly relativistic particle, we can expand the energy as powers in m^2/p^2 as

$$E = p\left[1 + \frac{m^2}{2p^2} - \frac{1}{8}\left(\frac{m^2}{p^2}\right)^2 + \ldots\right] \tag{75}$$

  Hence the rapidity reads

Kinematics XVI

$$
\begin{aligned}
y &= \frac{1}{2}\ln\left[\frac{p(1+\frac{m^2}{2p^2}+..)+p\cos\theta}{p(1+\frac{m^2}{2p^2}+..)-p\cos\theta}\right] \\
&= \frac{1}{2}\ln\left[\frac{\cos^2\frac{\theta}{2}+\frac{m^2}{4p^2}+...}{\sin^2\frac{\theta}{2}+\frac{m^2}{4p^2}+...}\right]
\end{aligned}
$$

where we used the fact that $p_{||} = p\cos\theta$, and the trigonometric identities $\cos\theta + 1 = 2\cos^2\theta/2$, and $1 - \cos\theta = 2\sin^2\theta/2$.

For massless particle, which is a good approximation for particles produced with very high energy, the rapidity reduces to the so-called **pseudo rapidity**:

$$
\eta = \frac{1}{2}\ln\left[\frac{p+p_{||}}{p-p_{||}}\right] = -\ln\left(\tan\frac{\theta}{2}\right) = \ln\left(\cot\frac{\theta}{2}\right)
$$

Kinematics XVII

which depends only on the angle θ, and hence can be easily measured. This is why particle distributions are often in terms of $dN/d\eta$ instead of dN/dy. Note that $\eta(\theta) = -\eta(180 - \theta)$, i.e. the pseudorapidity is odd about $\theta = 90$. In the table below we show the corresponding η for some values of the polar angle θ. A plot of η vs the angle θ in degree is shown in figure1.

$\theta(^0)$	0	10	20	30	45	90	180	170	160	150
η	∞	2.44	1.74	1.32	0.88	0	$-\infty$	-2.44	-1.74	-1.32

The Pseudorapidity is particularly useful in hadron colliders (such as the Tevatron or LHC), where the composite nature of the colliding protons means that interactions rarely have their centre of mass frame coincident with the detector rest frame, and where η is much easier to estimate than the rapidity.

Since the azimuthal angle is confined to planes perpendicular to the beam axis, say z-axis, $\Delta\phi$ is invariant under boosts along the z-axis. So in the plane $\eta - \phi$ plane we define the quantity

Kinematics XVIII

$$d = \sqrt{(\Delta\eta)^2 + (\Delta\phi)^2}$$

which represents the distance between two directions or particles (or jets) in the $(\eta - \phi)$ plane. Note that d is invariant under a boost transformation along the beam line.

Of course, the transverse momentum p_T is another invariant under under such boosts. For the process $A + B \to 1 + 2$, the conservation of momentum implies that transverse momentum of the final state 1 and 2 are equal and can be written in terms of the centre of mass energy and the scattering angle θ_{cm} in the centre of mass frame of the process (or the subprocess at parton level for if A and B are protons or antiprotons) as

$$p_T = p \sin\theta_{cm} = \frac{\sqrt{s}}{2} \sin\theta_{cm}$$

Kinematics XIX

So, often experiments measure the lab frame pseudo rapidities (which parametrize the scattering angle θ in the center of mass of a process) of the final state particles , their azimuthal angles, and their transverse momenta. For hadron final states usually, the results are presented as a lego plot in the $\eta - \phi - p_T$ coordinates of the jet.

[8] Here we used the fact that

$$E + p_{||} = m_{\perp} e^{y}, \text{ and } E_T = \sqrt{(E - p_{||})(E + p_{||})}$$

.

Proton-Proton colision I

- **Introduction**

At hadron colliders, such as the LHC, the collisions is between protons. However, protons are like messy bags of quarks and gluons, where each parton carries a fraction of the momentum of the proton with some probability distribution. But , in fact, this can be seen as an advantage over electron-positron colliders because now in every collision there will be a scanning over a range of energies. Unfortunately, this happens simultaneously and so we don't have control on the initial state (unlike in the lepton colliders), which means that we don't know the center of mass frame of the collision subprocess (at the parton level).

We have seen earlier, that the total cross section for proton-proton scattering is of oder 100 *mb*. Thus, one wonder how can one search for new physics which will have cross section of order **pico barn**, or even samller? It turns out that most of the debris of that collision goes down the beam line, also called **minimum bias events**, whereas the hard collisions, corresponding to

Proton-Proton colision II

the interactions at short distance, will be distinguished by scattering out with large angles and the presence of a hard scale Q which can be for example the invariant mass of final state particles, or the transverse momentum of the produced particles.

- **Parton kinematics**
  In the center of mass system, the proton 4-momenta are

$$P^{\mu} = E_b(1, 0, 0, 1), \quad \bar{P}^{\mu} = E_b(1, 0, 0, -1)$$

We denote k_A^{μ} and k_B^{μ} the momenta of the partons in the proton and 2 antiproton, respectively. Since the energies of the particles (both in the initial and final states) are much larger than the particle masses, I will take all the masses to be zero. Thus,

$$\hat{s} := (k_a + k_b)^2 \simeq 2k_a.k_b, \quad S := (P + \bar{P})^2 \simeq 2P.\bar{P} = 4E_b^2$$

Proton-Proton colision III

The momenta k_a and k_b are fractions of P_1 and P_2, i.e.

$$k_a = x_a P, \qquad k_b = x_b \bar{P} \quad 0 \leq x_{a,b} \leq 1$$

and so,

$$\hat{s}_{A,B} \simeq x_a x_b S$$

As it has been mentioned in the introduction, at the parton level, we don't know the center of mass energy of the collision because we don't know what x_a and x_b really are, and so the collision can be taken place at any boost along the z direction relative to the center mass frame of the protons. For this reason, as we shall see below, it is useful to use the lightcone coordinates, defined as

$$p^{\pm} := p^0 \pm p^z, \qquad \mathbf{p}_T = (p_x, p_y)$$

Proton-Proton colision IV

and we label a 4-momentum as $(p^+, p^-, \mathbf{p}_T)$. In these coordinates, the scalar product of two 4-vectors A and B reads

$$A.B = \frac{1}{2}(A^+B^- + A^-B^+) - \mathbf{A}_T.\mathbf{B}_T$$

Moreover, using the definition of the rapidity, we can write the momenta $p^\pm$ as

$$p^\pm = E_T e^{\pm y} \tag{76}$$

As we have shown earlier, under boosts in the z-direction with a velocity β, the rapidity shifts as

$$y \to y + \eta, \quad \eta := \frac{1}{2} \ln\left(\frac{1+\beta}{1-\beta}\right)$$

Proton-Proton colision V

Hence,

$$p^{\pm} \longrightarrow e^{\pm\eta} p^{\pm}$$

and we see that these coordinates transform rather nicely under boosts in the z-direction.

So, in this coordinates, the 4-momenta of the colliding protons read $P = \sqrt{S}(1,0,0,0)$ and $\bar{P} = \sqrt{S}(0,1,0,0)$, where we used the approximation that protons are massless in the Ultra relativistic collision. Thus,

$$(a) \qquad \text{incoming partons} : k_a = \sqrt{S}(x_a, 0, 0, 0), \quad k_b = \sqrt{S}(0, x_B, 0, 0)$$

$$(b) \qquad \text{Outgoing particles} : k_j = (E_{T_j} e^{y_j}, E_{T_j} e^{-y_j}, \mathbf{p}_{T_j})$$

Thus, from conservation of momenta we deduce

Proton-Proton colision VI

$$
\begin{aligned}
0 &= \sum_i \mathbf{p}_{T_j} \\
x_a &= \sum_j \frac{E_{T_j}}{\sqrt{S}} e^{y_j} \qquad (77)\\
x_b &= \sum_j \frac{E_{T_j}}{\sqrt{S}} e^{-y_j} \qquad (78)
\end{aligned}
$$

Therefore, if one could measure the rapidities and the transverse energies of all the particles that come out, one could determine the momenta fractions x_a and x_b of the partons.

Proton-Proton colision VII

- **The Cross section of** $p + p \to 1 + 2$:
  According to the parton model, the probability of finding a parton **a** with momentum fraction between x_a and $(x_a + dx_a)$ in a proton is

$$f_{a/p}(x_a; \mathcal{Q}^2)dx \tag{79}$$

where $f_{a/p}(x_a)$ are called the **parton distribution functions** (PDF), and $\mathcal{Q}$ is the energy scale which characterize the hard scattering (for more details see the appendix). The PDFs are nonperturbative quantities and no one knows how to compute them from first principle. They are determined from the data of the deep inelastic scattering, i.e. electron-proton scattering. Once the PDFs are determined at a given scale, the so called Dokshitzer-Gribov-Lipatov-Altarelli-Parisi (DGLAP) equation can be used to evolve them to any other scale. A non trivial fact about the PDFs is that once you determine them from some experiments, they can be used for all experiments.

Proton-Proton colision VIII

At the LHC, $\mathcal{Q}^2 \sim (TeV)^2$, and the cross section for the process $A + B \to 1 + ...n$ reads

$$d\sigma = \int_0^1 dx_a dx_b f_{A/p}(x_a) f_{b/p}(x_b) \left(\frac{|\mathcal{M}|^2}{2\hat{s}} \right) d\Phi_n$$

where $\mathcal{M}$ is the amplitude of the above process, and

$$\begin{aligned} d\Phi_n &= (2\pi)^4 \delta^{(4)}(x_a P + x_b \bar{P} - \sum_{j=1}^{n} p_j) \prod_{j=1}^{n} \frac{d^3 p_j}{2E_j (2\pi)^3} \\ &= \frac{(2\pi)^{(4-3n)}}{2^n} \delta^{(4)}(x_a P + x_b \bar{P} - \sum_{j=1}^{n} p_j) \prod_{j=1}^{n} dy_j d\mathbf{p}_{T,j} \end{aligned}$$

Integrating over the x's yields

Proton-Proton colision IX

$$d\sigma = \frac{f_{a/p}(x_a)f_{b/p}(x_b)}{S}\frac{|\mathcal{M}|^2}{\hat{s}}\delta(\sum \mathbf{p}_{T,j})\prod_{j=1}^{n} dy_j d\mathbf{p}_{T,j}$$

where now the fraction of momenta x_a and x_b are not free variables; they are given by the expressions (77) and (78), respectively.

Now let us restrict our self to the case of $n = 2$. Then, integrating over the transverse momentum of particle 2, we obtain

$$d\sigma = \frac{f_{a/p}(x_a)f_{b/p}(x_b)}{S}\frac{|\mathcal{M}(a+b\to 1+2)|^2}{\hat{s}} dy_1 dy_2 d^2\mathbf{p}_T \qquad (80)$$

where $\mathbf{p}_T \equiv \mathbf{p}_{T,1} = -\mathbf{p}_{T,2}$. In terms of the variables y_1, y_2 and $\mathbf{p}_T$, the Mandlestam variables read

Proton-Proton colision X

$$\hat{s} = x_a x_b S = E_{T,1}^2 + E_{T,2}^2 + 2E_{T,1}E_{T,2}\cosh\left(y_2 - y_1\right)$$
$$\hat{t} = (x_a P - p_1)^2 = -\left[\mathbf{p}_T^2 + E_{T,1}E_{T,2}e^{(y_2-y_1)}\right]$$

Note that both $\hat{s}$ and $\hat{t}$ depend p_T and on the difference of the rapidities $\Delta y = (y_2 - y_1)$. This dependence is somehow expected since the Mandelstam variables are Lorentz invariant, and, as we have shown before, Δy is invariant under boosts along the z axis. Furthermore, one can easily show that $dy_1 dy_2$ can be written as[9]

$$dy_1 dy_2 = d\bar{y}\, d(\Delta y)$$

Proton-Proton colision XI

where $\bar{y} = (y_1 + y_2)/2$ is the average of the rapidities of the two final state particles, which basically is a measure of how off is the centre of mass frame of the collision from the lab frame.

Since the spin averaged amplitude square will depend on $\hat{s}$ and $\hat{t}$, we can trade the variable Δy with $\hat{s}$, as follows

$$\begin{aligned} d\hat{s} &= 2E_{T,1}E_{T,2}\sinh\Delta y d\Delta y \\ &= 2E_{T,1}E_{T,2}\sinh\left[\cosh^{-1}\left(\frac{\hat{s} - E_{T,1}^2 - E_{T,2}^2}{2E_{T,1}E_{T,2}}\right)\right]d\Delta y \\ &\equiv \frac{1}{\mathcal{J}(\hat{s}, \mathbf{p}_T)}d\Delta y \end{aligned}$$

where

Proton-Proton colision XII

$$\mathcal{J}(\hat{s}, \mathbf{p}_T) = \frac{1}{\sqrt{((E_{T,1} - E_{T,2})^2 - \hat{s})\,((E_{T,1} + E_{T,2})^2 - \hat{s})}} \tag{81}$$

Thus,

$$dy_1 dy_2 = \mathcal{J}(\hat{s}, \mathbf{p}_T) d\bar{y} d\hat{s}$$

For the particular case where $M_1 = M_2$ so $E_{T,1} = E_{T,2} = E_T$, it is straightforward to show that

$$x_a = \sqrt{\frac{\hat{s}}{S}} e^{\bar{y}}, \quad x_b = \sqrt{\frac{\hat{s}}{S}} e^{-\bar{y}}, \quad \mathcal{J} = \frac{1}{\sqrt{\hat{s}(\hat{s} - 4E_T^2)}} \tag{82}$$

and so,

Proton-Proton colision XIII

$$\frac{d^2\sigma}{d^2\mathbf{p}_T} = d\bar{y} f_{a/p}(\sqrt{\hat{s}/S}e^{\bar{y}}) f_{b/p}(\sqrt{\hat{s}/S}e^{-\bar{y}}) \mathcal{J}(\hat{s}, \mathbf{p}_T) \frac{|\mathcal{M}|^2(\hat{s}, \mathbf{p}_T)}{\hat{s}} d\hat{s}$$

Note that the variable $\bar{y}$ enters only in the PDFs. So, If we only measure the transverse momentum, we can integrate over $\bar{y}$ and obtain

$$\frac{d^2\sigma}{d^2\mathbf{p}_T} = \frac{dL_{a,b}}{d\hat{s}}(\hat{s}) \mathcal{J}(\hat{s}, \mathbf{p}_T) \frac{|\mathcal{M}(a+b \to 1+2)|^2}{\hat{s}} d\hat{s} \qquad (83)$$

Proton-Proton colision XIV

where

$$\frac{dL_{a,b}}{d\hat{s}}(\hat{s}) = \frac{1}{S}\int d\bar{y} f_{a/p}(\sqrt{\hat{s}/S}e^{\bar{y}}) f_{b/p}(\sqrt{\hat{s}/S}e^{-\bar{y}})$$

which is called the **parton luminosity**, and it is given in units of cross section (e.g. pb or nb). An important observation is that for $x < 0.1$, the PDFs can be well approximated by power laws, and naively the dependence on $\bar{y}$ in product $f_{a/p}(\sqrt{\hat{s}}, \bar{y}) f_{b/p}(\sqrt{\hat{s}}, \bar{y})$ almost cancels. Hence, one expect that the rapidity distribution will be approximately flat up to values of x close to 0.1. This means that $\rho_{ab}(\hat{s})$ is a power law. Aplot of $dL_{a,b}/d\hat{s}$ is shown in figure3. By inspection, one finds that for $\sqrt{\hat{s}} \leq TeV$, the parton-parton luminosity for gluon-gluon and the quark-antiquark can be approximated as

$$\frac{dL_{a,b}}{d\hat{s}}(\hat{s}) \simeq \begin{cases} 2.5 \times 10^3 \left(\frac{TeV}{\sqrt{\hat{s}}}\right)^4 pb & : \mathrm{gg} \\ 4.5 \times 10^2 \left(\frac{TeV}{\sqrt{\hat{s}}}\right)^{3.3} pb & : q\bar{q} \end{cases}$$

Proton-Proton colision XV

where q here can be a up or down quark. Thus, the parton luminosity **falls** very rapidly with $\hat{s}$.

- **Some qualitative features of $\sigma(p + p \to 1 + 2)$**
  Let us specialise to the following cases:
  1. **Production of massless particles via resonance**
     We consider the production of the particles 1 and 2 via the process

$$a + b \to R \to 1 + 2 \tag{84}$$

where the resonance R has mass M_R and total width Γ_R. We assume that the masses of 1 and 2 are much smaller than the centre of mass energy of the process so that they can be considered to be massless. When $\hat{s} \simeq M_R$, we can use the narrow width approximation, i.e.

$$|\mathcal{M}|^2 \propto \frac{\text{product of coupling constant}}{(\hat{s} - M_R^2)^2 + M_R^2 \Gamma_R^2} \to K\delta(\hat{s} - M_R^2) \tag{85}$$

Proton-Proton colision XVI

where K is a the product of the coupling constant and the factor $\pi/M_X\Gamma_X$ which arises from approximating the propagator with a delta-distribution. Hence,

$$\frac{d^3\sigma}{d\bar{y}d^2\mathbf{p}_T} \propto d\bar{y}\int f_{a/p}(\sqrt{\hat{s}/S}e^{\bar{y}})f_{b/p}(\sqrt{\hat{s}/S}e^{-\bar{y}})\mathcal{J}(\hat{s},\mathbf{p}_T)\frac{\delta(\hat{s}-M_R^2)}{\hat{s}}d\hat{s}$$

which yields

$$d^3\sigma \propto \frac{d^2\mathbf{p}_T/M_R^2}{\sqrt{1-\frac{4\mathbf{p_T^2}}{M_R^2}}}\frac{1}{M_R^2}\left[d\bar{y}f_{a/p}(\sqrt{M_R^2/S}e^{\bar{y}})f_{b/p}(\sqrt{M_R^2/S}e^{-\bar{y}})\right] \quad (86)$$

Proton-Proton colision XVII

(a) The dependence of the number of events as function of $\mathbf{p_T^2}$ is

$$\frac{d^2\sigma}{d^2\mathbf{p}_T} \propto \frac{1}{\sqrt{1-\frac{4\mathbf{p_T^2}}{M_R^2}}}$$

which sharply peak at $\mathbf{p_T} = \mathbf{M_R}/\mathbf{2}$, also called the **Jacobian peak**. Note that the conservation of energy in the parroting process requires that $E_T \leq \sqrt{\hat{s}}/2$, and so the maximum value of the transverse momentum when $\sqrt{\hat{s}} = M_R$ is excepted to be $M_X/2$ for massless particles. In practice, the peak is is smeared by finite width of the particle X and QCD radiation.

(b) The distribution of the number of events versus the rapidity reads

$$\frac{d\sigma}{d\bar{y}} \propto f_{a/p}(\sqrt{M_R^2/S}e^{\bar{y}})f_{b/p}(\sqrt{M_R^2/S}e^{-\bar{y}})$$

which, as we stated earlier, will be **flat** for $x \leq 10^{-1}$ (where the PDFs started to have approximate power law behaviour), which corresponds to

Proton-Proton colision XVIII

$$M_R e^{\bar{y}} \sim \frac{E_{cm}}{10}$$

at which the distribution **falls very quickly** to zero.

s

kip (c) The total cross section can be obtained by integrating (86) over the transverse momentum(which gives a factor of $\pi/2$) and the rapidity, which parametrically yields

$$\sigma_{\text{tot}} \sim \frac{1}{M_R^2} \rho_{ab}(M_R^2/S) \sim \frac{1}{M_R^2} \left(\frac{E_{cm}}{M_R}\right)^{2\alpha}$$

Thus,

$$\sigma_{\text{tot}} \sim \begin{cases} \frac{1}{M_R^6} & : \quad a = b = \text{gluon} \\ \frac{1}{M_R^{5.3}} & (a, b) = (u, \bar{u}), (d, \bar{d}) \end{cases}$$

Proton-Proton colision XIX

The important result is that, the cross section for the production of SM particles via resonance at the LHC drops as large power of M_R: roughly between $1/M_R^5$ and $1/M_R^6$, which is due to the PDFs. This means that, at the LHC it is important to cover very large range of cross sections because a factor of 10 in the mass will correspond to about a factor of 10^6 in the rate.

2. **Production of heavy particles:**
Now let us assume that the particles 1 and 2 are heavy enough that their masses can not be neglected. Also, assume that they have the same mass M. Then, the differential cross section for the pair production of these particles can be re-written as

$$d^3\sigma \propto \frac{d^2\mathbf{p}_T/M^2}{\frac{\hat{s}}{M^2}\sqrt{\frac{\hat{s}}{M^2}\left(\frac{\hat{s}}{M^2}-\frac{4E_T^2}{M^2}\right)}}\rho_{a,b}(\hat{s})|\mathcal{M}(a+b\rightarrow 1+2)|^2 d(\hat{s}/M^2)$$

Now, notice that because first factor falls like $1/\hat{s}$ and the parton luminosity goes like $1/\hat{s}^2$, the collision is dominated by values of $\hat{s}=2E_T^{(\min)}$. This means

Proton-Proton colision XX

rate for the collision is dominated by the smaller values of p_T of the produced particles.

At the LHC, heavy particles are primarily produced at threshold.

Proton-Proton colision XXI

- **Top production and detection at LHC**

  The initial hard scattering that yields $t\bar{t}$ final state involves either gluons or a quark and an antiquark.

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[9] Recall that under the transformation

$$\begin{cases} x = x(u, \upsilon) \\ y = y(u, \upsilon) \end{cases}$$

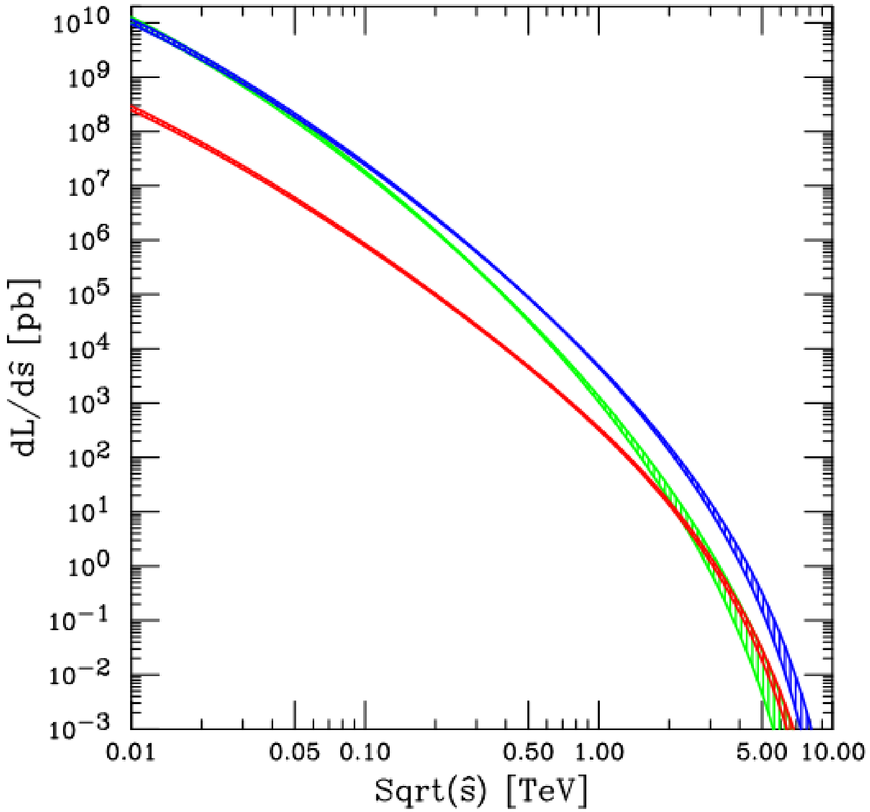
the differential element $dxdy$ is related to $dud\upsilon$ as

$$dxdy = |\frac{\partial(x, y)}{\partial(u, \upsilon}|dud\upsilon$$

where

$$|\frac{\partial(x, y)}{\partial(u, \upsilon}| = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial \upsilon} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial \upsilon} \end{vmatrix}$$

is the **Jacobian** of the transformation. Generalisation to more than two variables is straightforward.



Green $= gg$, Blue $= gq + g\bar{q} + qg + \bar{q}g$, Red $= q\bar{q} + \bar{q}q$ $(q = d + u + s + c + b)$.

Figure: Parton luminosity

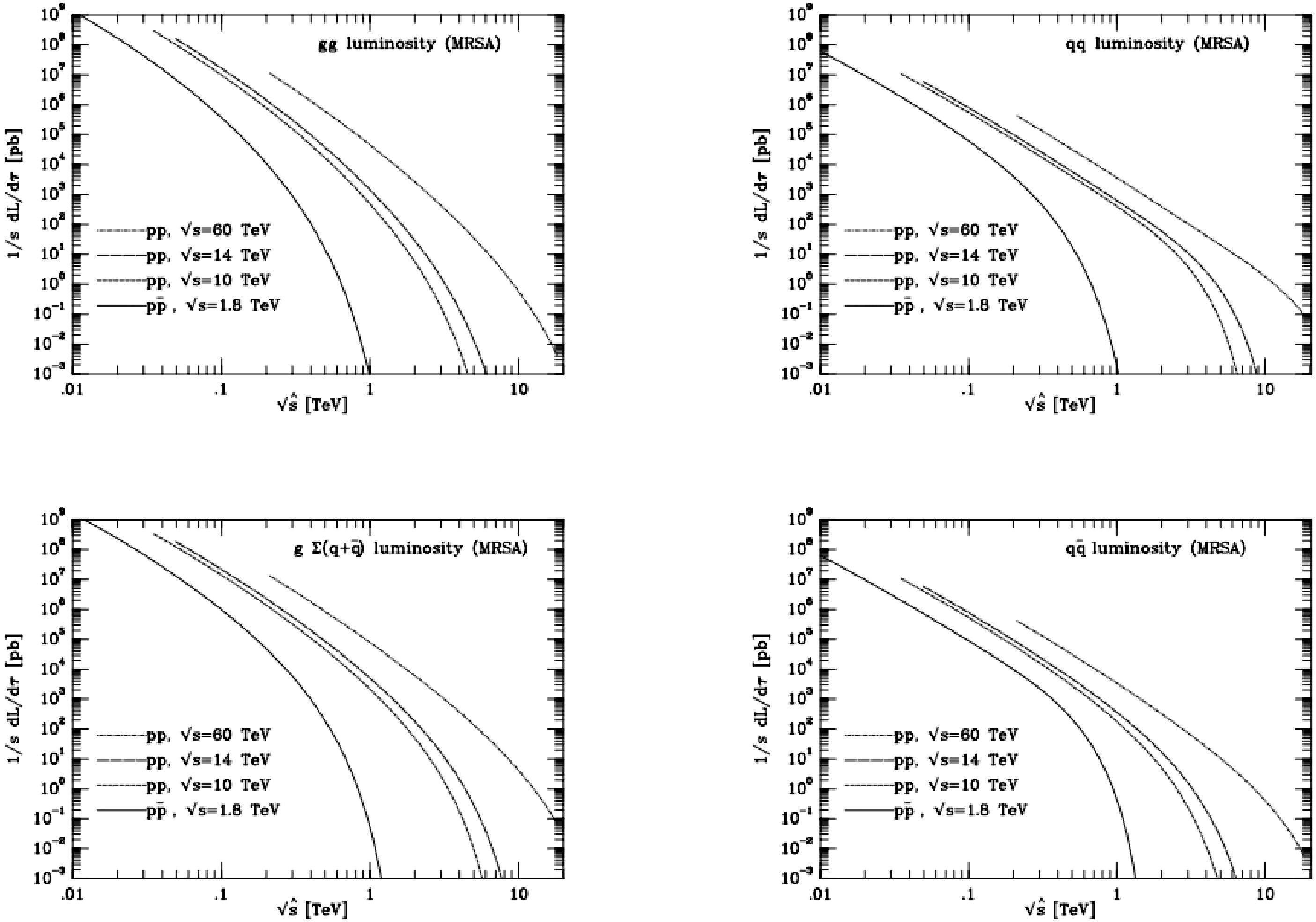


Figure: Parton luminosity

Scattering in Scalar QED I

- **The Lagrangian**:
  Let ϕ be a complex scalar field representing a charged particle interacting with $U(1)$ gauge field A_μ. The Lagrangian is given by

$$\begin{aligned}\mathcal{L} &= -\frac{1}{4}F_{\mu\nu}^2 - |D_\mu\phi|^2 - m^2|\phi|^2 \\ &= -\frac{1}{4}F_{\mu\nu}^2 - gA_\mu J^\mu + g^2 A_\mu A^\mu |\phi|^2\end{aligned}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the gauge field strength, and

$$D_\mu\phi = \partial_\mu\phi + igA_\mu\phi, \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

and

$$J_\mu = i\left(\phi^*\partial_\mu\phi - \partial_\mu\phi^*\phi\right)$$

Scattering in Scalar QED II

- **Cross section for the process** $\phi(p_1)\phi(p_2) \to \phi(p_3)\phi(p_4)$:
  There are two diagrams that contribute to this process:
  (a) t-channel diagram:

$$i\mathcal{M}_t = (-ig)\left(p_1^\mu + p_3^\mu\right) \frac{-i\left[\eta_{\mu\nu} + (1-\xi)\frac{k_\mu k_\nu}{k^2}\right]}{k^2} (-ig)\left(p_2^\mu + p_4^\mu\right)$$

  (b) u-channel diagram

Drell-Yan scattering: Production of l^+l^- I

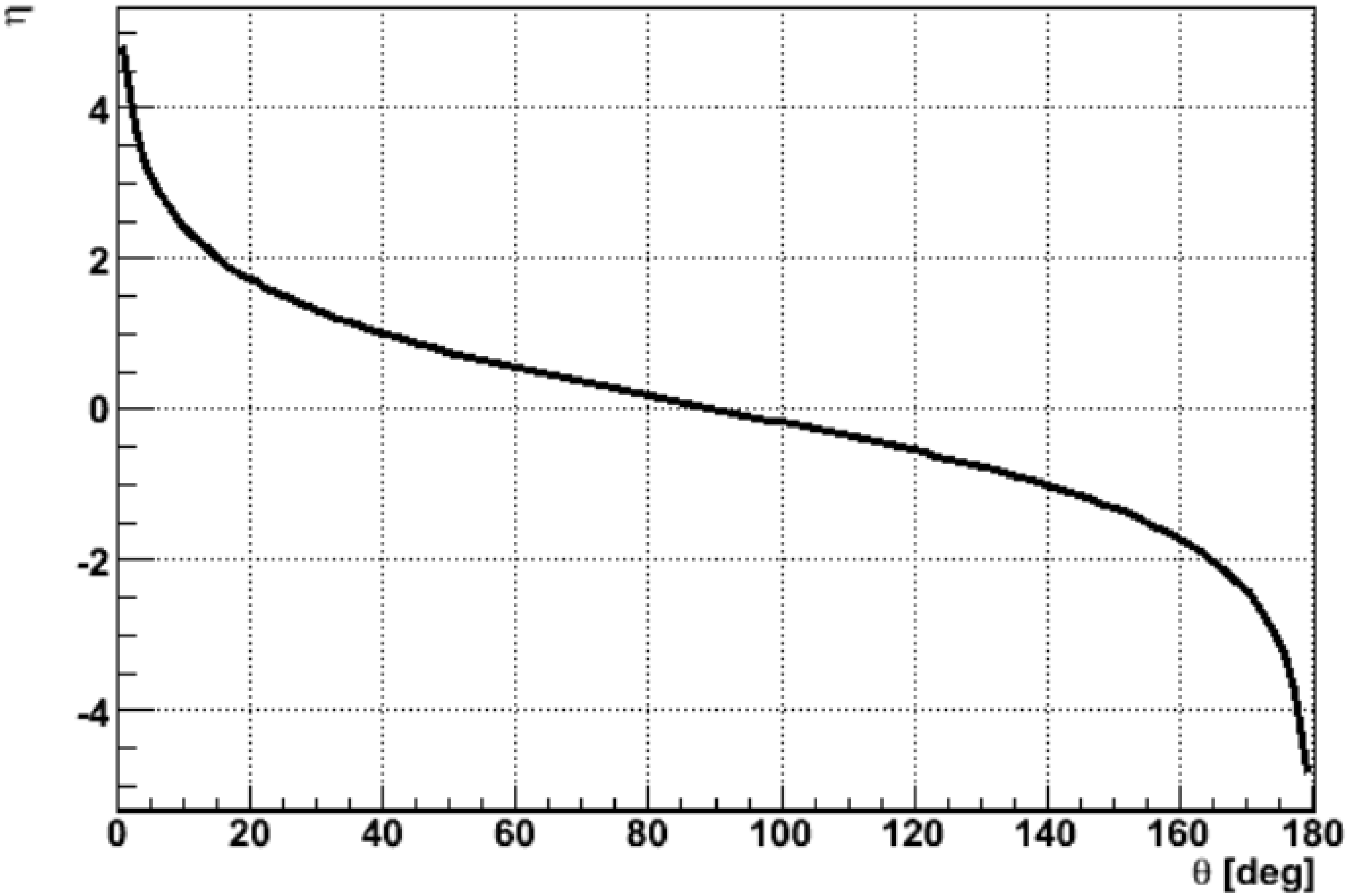


Figure: Pseudorapidity

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