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The motion of a Rocket

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Abstract. These are my notes on some selected topics in undergraduate Physics.

Contents

1 Introduction

In **1903**, **Konstantin Tsiolkovsky** (1857- 1935), a Russian Physicist and school teacher, published "The Exploration of Cosmic Space by Means of Reaction Devices", in which he presented all the basic equations for rocketry. He determined that liquid fuel rockets would be needed to get to space and that the rockets would need to be built in stages. He concluded that oxygen and hydrogen would be the most powerful fuels to use. He had predicted in general how, 65 years later, the Saturn V rocket would operate for the first landing of men on the Moon.

Robert Goddard, an American university professor who is considered "the father of modern rocketry," designed, built, and flew many of the earliest rockets. In **1926** he launched the world's first liquid fueled rocket.

The German scientist **Hermann Oberth** independently arrived at the same rocketry principles as Tsiolkovsky and Goddard. In **1929**, he published a book entitled "By Rocket to Space", [1] that was internationally acclaimed and persuaded many that the rocket was something to take seriously as a space vehicle.

One of the leading figures in the development of pre-war Germany's rocket program and the development of the V2 missile is **Von Braun** (1912-1977). He entered the United States after the war and became a naturalized citizen. He worked on the development of intercontinental ballistic missiles and led the development team that launched Explorer 1. Von Braun was the chief architect and engineer of the Saturn V Moon rocket[2].

2 Types of Rockets

A rocket is an engine that produces a force, a thrust, by creating a high velocity output without using any of the constituents of the atmosphere in which the rocket is operating. This means that it can operate in any part of the atmosphere and outside the atmosphere which makes it ideal for space propulsion[2]. The thrust is produced because the rocket engine must have exerted a force on exhaust material and an equal and opposite force, the thrust, therefore, exerted on the rocket.

There are two basic types of rocket engines:

- **Chemical Rockets:**

[1] Actually this book was based on his dissertation for the University of Heidelberg, which was rejected for being too speculative.

[2] On July 20, 1969, American astronaut Neil Armstrong set foot on the Moon. It was the first time in history that humans had touched another world.

[2] Means pushing or driving forward.

In such engines, a fuel (e.g. liquid hydrogen (LH_2) or Kerosene) and an oxidizer (liquid oxygen (LO_2)) are usually supplied to the combustion chamber of the rocket. The chemical reaction between the fuel and the oxidizer produces a high pressure and temperature in the combustion chamber and the gaseous products combustion can be expanded down to the ambient pressure, which is much lower than the combustion chamber pressure, giving a high velocity gaseous efflux from the rocket engine.

Chemical rockets are unique in that the energy required to accelerate the propellant comes from the propellant itself, and in this sense, are considered energy limited. Thus, the attainable kinetic energy per unit mass of propellant is limited primarily by the energy released in chemical reaction; the attainment of high exhaust velocity requires the use of high-energy propellant combinations that produce low molecular weight exhaust products. Currently, propellants with the best combinations of high energy content and low molecular weight seem capable of producing specific impulses (see later) in the range of 400 to 500 seconds or exhaust velocities of $13,000 - 14,500 ft/sec$[1] is the universal gas constant, $\gamma \simeq 1.2$ is the ratio of specific heat, $T_{\rm ooc}$ is the combustion chamber temperature, p_{atm} and p_{occ} are the atmospheric and the combustion pressures, respectively.

- **Non-Chemical Rockets:**

  In a non-chemical rocket, the high efflux velocity from the rocket is generated without any chemical reaction taking place. For example, a gas could be heated to a high pressure and temperature by passing it through a nuclear reactor and it could then be expanded through a nozzle to give a high efflux velocity.

There have been many types of rockets developed by NASA. The Mercury Redstone 3 rocket carried the spacecraft of America's first astronaut, Alan Shephard, into space. The Atlas 6 rocket carried John Glenn's spacecraft into Earth's orbit, making him the first American to ever orbit the Earth. The Titan rocket carried the Gemini 12 mission into space. Titan series rockets carried many Gemini missions into space.

The Saturn V[2] launch vehicle was used for Apollo flights to the Moon (from 1967 to the end of 1972). The rocket was 364 feet tall (taller than the Statue of Liberty) and included the spacecraft and three rocket stages. Each rocket stage pushed the spacecraft farther and farther from Earth. The Saturn V flew ten missions to the

[1] The thermal rocket exhaust velocity is given by

$$V_e = \sqrt{\frac{2\gamma}{\gamma - 1}\frac{\mathcal{R}T_{\rm occ}}{\bar{\mathcal{M}}}\left[1 - \left(\frac{p_{atm}}{p_{\rm occ}}\right)^{\frac{\gamma-1}{\gamma}}\right]} \tag{2.1}$$

where $\bar{\mathcal{M}}$ is the average molecular weight of the exhaust, $\mathcal{R} = 8314.3\ Joules/mole.K$

[2] The "V" designation originates from the five powerful F-1 engines that powered the first stage of the rocket.

Moon, three un-piloted and seven piloted. (Apollo XIII was an unsuccessful mission that returned safely to Earth).

The Soviets launched the Soyuz spacecraft into Earth orbit to meet the American Apollo spacecraft. The two spacecraft met in space, proving that such a rendezvous was possible. When the two crafts connected, American astronauts and Soviet Cosmonauts marked the meeting with an historic handshake. The space shuttle has three main rocket engines and a large external fuel tank. Two additional rockets are needed to assist the shuttle in its journey into space.

3 Derivation of the equation of a rocket

Consider a rocket of mass $M(t)$, moving at velocity $\boldsymbol{v}(t)$ at the instant t and subject to external forces $\boldsymbol{F}_{ext}$ (typically gravity or drag), and let $\boldsymbol{v}$ be the velocity of a small amount of gas of mass $|dM|$ expelled by the rocket in the instant $(t{+}dt)$. the momentum of the rocket at instant t is

$$\boldsymbol{P}^{(in)} = M\boldsymbol{v} \tag{3.1}$$

At the instant $(t + dt)$, a small amount of gas of mass $|dM|$ is expelled by the rocket with velocity $\boldsymbol{v}'$ with respect to a stationary observer on the ground, and the rocket will have mass is $(M - |dM|)$ and velocity $(\boldsymbol{v} + d\boldsymbol{v})$. Thus, momentum of the system is

$$\boldsymbol{P}^{(fin)} = (M - |dM|)\,(\boldsymbol{v} + d\boldsymbol{v}) + |dM|\boldsymbol{v}' \tag{3.2}$$

Thus, Newton's second law reads,

$$\boldsymbol{F}_{ext}dt = |dM|\,(\boldsymbol{v}' - \boldsymbol{v}) + Md\boldsymbol{v} \tag{3.3}$$

For the case of the rocket, $|dM/dt| = -dM/dt$, and so we get

$$M\frac{d\boldsymbol{v}}{dt} = \boldsymbol{F}_{ext} + \boldsymbol{V}_e\dot{M} \tag{3.4}$$

Here, $\vec{V}_e = (\boldsymbol{v}' - \vec{v})$ is the velocity of $|dM|$ relative to the the rocket. This expression is valid when $\dot{M} < 0$ (mass loss, such as the case of rocket) as well when $\dot{M} > 0$ (mass mass gain). The term $\boldsymbol{V}_e\dot{M}$ is called the thrust, and can be interpreted as an additional force acting on the rocket due to gas expulsion. Equation (3.6) can be also written in the familiar form

$$\frac{d\,(M\vec{v})}{dt} = \vec{F}_{ext} + \vec{v}'\dot{M} \tag{3.5}$$

which shows that, for systems involving variable mass, the usual expression $d\left(M\vec{v}\right)/dt = \vec{F}_{ext}$ is only valid when the final (initial) velocity of the expelled (captured) mass, $\boldsymbol{v}'$, is zero. Along the flight path, the above equation reads

$$M\frac{dv}{dt} = F_t - V_e\dot{M} \tag{3.6}$$

where F_t is the component of F_{ext} along the direction of $\boldsymbol{v}$ (i.e. tangent to the path), and v and V_e are the magnitudes of $\boldsymbol{v}$ and V_e. If F_t is known, this equation can be integrated in time to yield an expression for the velocity as function of time.

Another way to derive the rocket equation:

Let $\vec{f}$ be the impulsive force that M exerts on $|dM|$ during the short time interval dt. Thus, from point of view of $|dM|$, the impulse $\boldsymbol{f}dt$ is equal to the change of momentum of $|dM|$,

$$\boldsymbol{f}dt = |dM|(\boldsymbol{v}' - \boldsymbol{v}) \tag{3.7}$$

From point of view of M, in addition to the external forces $\boldsymbol{F}_{ext}$, by Newton's third law, M will experience a force $-\boldsymbol{f}$, exerted by $|dM|$. Hence, impulse equation for M is

$$\boldsymbol{F}_{ext}dt - \boldsymbol{f}dt = M(\boldsymbol{v} + d\boldsymbol{v}v) - M\boldsymbol{v} \tag{3.8}$$

By combining (3.7) and (3.9), we get

$$M\frac{d\vec{v}}{dt} = \vec{F}_{ext} + \vec{V}_e\dot{M} \tag{3.9}$$

which is the exactly the equation in (3.6).

In the absence of external forces, the integration of (3.6) gives

$$M(t) = M_0 e^{-\frac{\Delta v}{V_e}} \tag{3.10}$$

where M_0 and $M_f = M(t_f)$ are the mass of the rocket at t_0 and t_f respectively. Equation (3.10) is known as the **rocket equation**, or the **Tsiolkovsky rocket equation**, which gives the mass of the rocket as a function of the initial mass M_0, Δv, and V_e. which is known as the **rocket equation**, or the **Tsiolkovsky rocket equation**, which gives the mass of the rocket as a function of the initial mass M_0, Δv, and V_e.

In general, the external force consists of the force of gravity $M\vec{g}$ and the drag force $\vec{D}$. Then (3.6) reads

$$\frac{dv}{dt} = -\frac{V_e}{M}\dot{M} - g\sin\theta - \frac{D}{M} \tag{3.11}$$

where θ is angle that the velocity vector makes with respect to the horizontal. By integrating (3.11) between some initial time t_0 and a final time t_f, we obtain

$$\Delta v = \Delta v|_{space} + \Delta v|_{gravitational} + \Delta v|_{drag} \tag{3.12}$$

where

$$\Delta v|_{space} = V_e \ln \frac{M_0}{M_f} \tag{3.13}$$
$$\Delta v|_{gravitational} = -\int_0^{t_f} g \sin\theta dt$$
$$\Delta v|_{drag} = -\int_0^{t_f} \frac{D}{M} dt$$

The effect of the drag force D is harder to quantify. It depends on the size, shape, and the speed of the vehicle, the Mach number and the local properties of the atmosphere through which the the vehicle is passing. It is given by

$$D = \frac{1}{2}\rho v^2 A C_D \tag{3.14}$$

Here ρ is the air density, C_D is the drag coefficient, and A is the cross sectional area of the rocket. The air density changes with altitude z, and may be approximated (perfect gas approximation) by

$$\rho = \rho_0 e^{-z/H} \tag{3.15}$$

where $H \simeq 8000\ m$ is the so-called **scale height** of the atmosphere, and ρ_0 is the density at sea level. It turns out that the differential equation for the velocity can not be integrated explicitly, and needs to be integrated numerically. however, to see the importance of the drag versus gravity, we estimate the ratio D/Mg. For a rocket of mass of 12000 kg with a cross section of $A \sim 1\ m^2$ and C_D, the drag force is maximal for $\rho \simeq 0.25\ kg/m^3$ and speed $v \simeq 700\ m/s$. Thus,

$$\frac{D}{Mg} \simeq 0.02 \tag{3.16}$$

which shows that the drag force is only about 2% of the gravity force, and so it is reasonable to ignore it in the first approximation. Thus, the gravitational losses are by far the largest for the space velocity increment.

If the rocket is launched vertically from rest with a constant thrust, and assuming that g is constant[1], then after integration the velocity is given by

$$\Delta v(t) = V_e \left[\ln \frac{1}{\mu} - \frac{1-\mu}{n} \right] \tag{3.17}$$

where $\mu = M/M_0$ and $n = -V_e \dot{M}/gM_0$ is thrust induced acceleration in units of g, also called the number of **gees**[2]. Note that we have assumed that $t = 0$, $V_e\dot{M} > M_0 g$, otherwise the rocket will sit on the pad, burning fuel until the remaining mass satisfies

[1] Assuming g to be constant may not be always as a justifiable assumption.

[2] Note that $\dot{M}$ is constant

this requirement. By integrating the velocity equation, and taking $v_0 = 0$, we find that the trajectory of the rocket is given by

$$z = \frac{V_e^2}{gn}\left[1 - \mu \ln\frac{1}{\mu} - \mu\frac{(1-\mu)^2}{2n}\right] \tag{3.18}$$

It is often used in practice to characterize the performance of a rocket engine by the so-called specific impulse, I_{sp}, defined as the ratio of the magnitude of the thrust to the propellant weight flow rate:

$$I_{sp} = \frac{V_e|\dot{M}|}{|\dot{M}|g} = \frac{V_e}{g} \tag{3.19}$$

Thus, the greater the specific impulse, the greater the net thrust and performance of the engine. Typical values of I_{sp} are around 300 s. Values of $I_{sp} \simeq 370$ are obtained by using **hydrogen**[1] as fuel and burning it either **oxygen** or **fluorine**. Value of I_{sp} up to 500 s can be achieved for higher energy fuels.

4 Optimizing a Single-Stage Rocket

By denoting M_L, M_s, and M_p to be the masses of payload (the section of the rocket that carries the cargo to be delivered), structure (such as the engine and empty tank), and propellant, respectively, we define the following coefficients

- Payload fraction

$$\Pi = \frac{M_L}{M_0} \tag{4.1}$$

- The structural coefficient

$$\epsilon = \frac{M_s}{M_s + M_p} \tag{4.2}$$

Smaller values of ϵ are highly desirable, since small structural mass means that more payload can be taken. For V-2 ballistic missile $\epsilon \simeq 0.3$, which is considered to be huge value. Launch vehicles posses much smaller structural coefficient. For example, the 1^{st} stage of the Saturn-V moon rocket had $\epsilon \simeq 0.07$. Generally, ϵ strongly depends on the density of the propellant: more dense propellants needs smaller tanks. This is why low density is one of the highest disadvantages of the LOX/LH2 propellants[3]. However, solid fuel motors, in spite of their high densities, needs thick castings, and hence their structural coefficients are large as well.

[1]The hydrogen is a preferred fuel because when it burns with oxygen it gives H_2O which has small average molecular weight, and hence a larger exhaust velocity (see the expression of V_e in the footnote in page 3.)

[3]It has very low density of only 0.28 kg/m^3 and needs large tanks.

The mass ratio M_f/M_0 can be written as

$$\begin{aligned}\frac{M_f}{M_0} &= \frac{M_L + M_s}{M_L + M_s + M_p} = 1 - \frac{M_s + M_p}{M_0}\frac{M_p}{M_s + M_p} \\ &= 1 - (1-\Pi)(1-\epsilon) \\ &= \epsilon + (1-\epsilon)\,\Pi\end{aligned} \tag{4.3}$$

Using (4.2) and (4.1), we can cast $\Delta v|_{space}$ as

$$\Delta v|_{space} = -V_e \ln\left[\epsilon + (1-\epsilon)\,\Pi\right] \tag{4.4}$$

For a given value of ϵ and the ratio $\beta = \Delta v|_{space}/V_e$, the allowed payload mass fraction that can be carried on board the vehicle is given by

$$\Pi = \frac{e^{-\beta} - \epsilon}{(1-\epsilon)} \tag{4.5}$$

and consequently, the propellant (fuel) mass fraction is

$$\begin{aligned}\frac{M_p}{M_0} &= (1-\epsilon)\,(1-\Pi) \\ &= \left(1 - e^{-\beta}\right)\end{aligned} \tag{4.6}$$

In order to reach a LEO [1], requires an orbital velocity of approximately 7700 m/s[1]. This means that the rocket must achieve a velocity increment of 9000 m/s where the extra velocity is needed to overcome gravity and drag[2]. if we suppose that the rocket can produce exhaust jets at velocities as high as $V_e \sim 4500\ m/s$, then $\beta = 2$, and hence

$$\frac{M_p}{M_0} = 87\% \tag{4.7}$$

Thus the the vehicle must have at least 87% propellant mass fraction. If we assume that the structural mass is about 10% of the propellant, which means that $\epsilon = 0.09$, then we find that

$$\Pi = 0.048 \tag{4.8}$$

[1]The lowest altitude where a stable orbit can be maintained, is at an altitude of about 200 km.

[1]For a circular orbit, the velocity is constant throughout the orbit, and given by

$$v_{circ} = \sqrt{\frac{\mu}{r_{circ}}}$$

where μ is a gravitational parameter and r_{circ} is the radius of the circular orbit to the planetary center. For the Earth, $\mu_E = 398600\ km^2/s^2$, and has radius $R_E = 6378\ km$. Thus a 185 km altitude circular LEO orbit has a velocity of approximately 7.8 km/s.

[2]The losses due to gravity and drag are of about $25\% - 30\%$. To reach moon or Mars, the required Δv is more than 10 km/s.

which is a very small fraction of the rocket weight. This shows the limitation in the payload mass that can be carried. One possible solution is to drop off the empty tank plus engine once the fuel is burnt. Although the thrust is the same, the mass of the rocket is smaller, and so the acceleration will be greater. However, with current technology and fuels, a single stage rocket to orbit (SSTO) is still not possible.

In fig.1, we plot $\Delta v|_{space}/V_e$ v.s Π for different values of ϵ. We can see that, without payload and for a reasonable value of the structural coefficient $\epsilon \simeq 0.1$, the rocket can (theoretically) achieve a velocity increment $\Delta v|_{space} \simeq 2.8V_e$. For LOX/kerosene propellant with $V_e \simeq 3400\ m/s$ it .....

5 Multi-stages Rocket

Multistage rocket[1] is a series of individual vehicles or stages, each with its own structure, tanks and engines. Multistage rocket permits to achieve higher velocities and carry more payload for space vehicles[3]. In this case, the initial mass in the equation is the total mass of the rocket at ignition, and the final mass is the total mass of the rocket at burnout (prior to discarding expendable stages). After the propellant is fully consumed in a particular stage, the remaining empty mass (tank plus its engine) of that stage is dropped from the vehicle and the propulsion system of the next stage is started. The last stage carries the payload.

To show that in a multi-stages rocket, the payload fraction can be higher than a single one, we consider an example of two stage vehicle. We again assume that each empty tank plus its engine weighs 10% of the propellant it carries, and the exhaust velocity is the same in the stages and equal to 4500 m/s. The required $\Delta v|_{space}$ is now divided into two Δv's of 4.5 km/s each. Then we have

- In the first stage, if M_{f1} is the mass after burn out, we have

$$M_{f1} = e^{4.5/4.5} M_0 = 0.368 M_0 \tag{5.1}$$

  where M_0 is the total initial mass of the rocket. The fuel burnt will be

$$M_{p_1} = (M_0 - M_{f1}) = 0.632 M_0 \tag{5.2}$$

  Hence the weight of the tank and the engine to be dropped will be $M_{s_1} = 0.0632 M_0$, leaving an initial mass for the second stage,M_{f2}, of

$$M_{02} = 0.368 M_0 - 0.063 M_0 = 0.295 M_0 \tag{5.3}$$

- In the second stage, the mass after complete burn out is

$$M_{f2} = e^{-1} M_{02} = 0.109 M_{02} \tag{5.4}$$

[1]The idea of multi-staging of a rocket is credited to Konstantin Tsiolkovsky .

[3]It also improve performance for long-range ballistic missiles.

The fuel burnt in this stage will be

$$M_{p_2} = (M_{02} - M_{2f}) = 0.186 M_0 \tag{5.5}$$

So the weight of the tank plus engine will be $M_{s_2} = 0.019 M_0$, leaving for the payload

$$M_L = (0.109 - 0.019)\, M_0 = 0.09 M_0 \tag{5.6}$$

Although it is still small but about twice the size of the payload obtained for the single stage rocket.

As we stated earlier, multistage rockets are able to reach orbital velocity because they discard structural weight during boost. This can be seen by again considering a two-stage rocket, where M_{s_i} and M_{p_i} denote the masses of the structure and propellant at the i^{th} stage (here $i = 1, 2$), respectively, and M_L is the pay load. If, for simplicity, assume that the exhaust velocity to be V_e for both stages, and that the tank is dropped off with zero velocity relative to the remaining rocket[1], then the final velocity is given by

$$\Delta v^{(2-staging)}|_{space} = V_e \ln \left(\frac{M_{01}}{M_{f1}} \right) \left(\frac{M_{02}}{M_{f2}} \right) \tag{5.7}$$

where $M_{01} = (M_{s_1} + M_{s_2} + M_{p_1} + M_{p_2} + M_L)$ is the initial mass of the rocket at the take off, $M_{02} = (M_L + M_{s_2} + M_{p_2})$ is the mass of the second-stage vehicle after structure of the first stage is dropped off, $M_{f1} = (M_{s1} + M_{s_2} + M_{p_2} + M_L)$ and $M_{f2} = (M_{s_2} + M_L)$ are the masses of the rocket just after the complete burn out of the fuel in first and second stage, respectively. The velocity increment (5.8) can be written as

$$\Delta v^{(2-staging)}|_{space} = \Delta v^{(single-rocket)}|_{space} + \delta^{(2-staging)}\,(\Delta v) \tag{5.8}$$

where

$$\Delta v^{(single-rocket)}|_{space} = V_e \ln \left(\frac{M_0}{M_f^{(single-rocket)}} \right) \tag{5.9}$$

and

$$\delta^{(2-staging)}\,(\Delta v) = V_e \ln \left(\frac{M_f^{(single-rocket)}}{M_{f1}} \right) \left(\frac{M_{02}}{M_{f2}} \right) \tag{5.10}$$

Here $M_f^{(single-rocket)} = (M_{s1} + M_{s_2} + M_L)$ is the final mass of a single-stage rocket. Let us now assume that $M_{s_1} = M_{s_2} = M_S/2$, and $M_{p_1} = M_{p_2} = M_P/2$. Since the

[1] If it is dropped off with some non zero velocity relative to the rocket, then there will be an extra gain in the initial velocity of the next stage.

propellant mass is the dominant contribution to the total mass of the rocket, we can expend the second term in (5.8) in powers of M_S/M_P and M_L/M_P, and obtain

$$\begin{aligned}
\delta^{(2-staging)}(\Delta v) &= V_e \ln\left[1 + \frac{M_S}{M_S + 2M_L}\left(1 - 2\frac{(M_S + M_L)}{M_P} + 4\frac{(M_S + M_L)^2}{M_P^2}\right)\right] \\
&= V_e \ln\left[1 + \frac{M_S}{M_S + 2M_L}\left(1 - 2\frac{(M_0 - M_P)}{M_P} + 4\frac{(M_0 - M_P)^2}{M_P^2}\right)\right] \quad (5.11)
\end{aligned}$$

From the above expression we see that $\delta(\Delta v)$ is positive provided that $M_P/M_0 > \frac{2}{3}$, which is the case in almost all rocket.

To get an idea an idea about the magnitude of $\delta(\Delta v)$, we consider a rocket of total mass $M_0 = 100$ tonnes, carrying a payload of mass $M_L = 1$ tonnes, and with a structural mass of about 10% of the rocket mass. We assume that the engines develop a constant exhaust velocity of $V_e = 2700\ m/s$. In this case, equations (5.9) and (5.10) yield

$$\Delta v^{(single-rocket)}|_{space} = 2700\ m/s \ln\left[\frac{10 + 89 + 1}{10 + 1}\right] \simeq 5960\ m/s \quad (5.12)$$

$$\delta^{(2-staging)}(\Delta v) = 2700\ m/s \ln\left[\left(\frac{10 + 1}{10 + 44.5 + 1}\right)\left(\frac{5 + 44.5 + 1}{5 + 1}\right)\right] \simeq 1382\ m/s$$

Thus, the final velocity of this 2-stage rocket is

$$\Delta v^{(2-staging)}|_{space} = 7342\ m/s \quad (5.13)$$

If the above rocket is divided into three stages with the fuel and the structural mass being shared equally amongst the three steps, then it is straight forward to show that the extra stage improve the velocity by another 749 m/s. Thus, that the final velocity achieved by the three-stage rocket is $\Delta v^{(3-staging)}|_{space} = 8092\ m/s$.

For the general case of an n-stage rockets, we denote by

- M_{0i}: The total initial mass of the i^{th} stage prior to firing including payload mass, that is the mass of $i, i+1, ..., n$ stages.
- M_{pi}: The mass of the propellant in the i^{th} stage.
- $M_{B_i} := M_{0i} - M_{pi}$ is the burnt out mass at the i^{th} stage.
- M_{si}: The structural mass of the i^{th} stage including the mass of its engine.
- $M_{L_i} := M_{0(i+1)}$.
- M_L: The payload mass.

For the i^{th} partial rocket, we define the structural coefficient ϵ_i, the payload ratio λ_i, and the mass ratio R_i as follow

- **Structural coefficient** : $\epsilon_i = \frac{M_{si}}{M_{0i}-M_{0,i+1}} = \frac{M_{si}}{M_{si}+M_{pi}}$

- **Payload ratio**: $\lambda_i = \frac{M_{L_i}}{M_{0i}-M_{0(i+1)}} = \frac{M_{0,i+1}}{M_{0i}-M_{0,i+1}}$

- **Mass ratio**: $R_i = \frac{M_{0i}}{M_{B_i}} = \frac{M_{0i}}{M_{0i}-M_{pi}} = \frac{1+\lambda_i}{\epsilon_i+\lambda_i}$

- **Total payload ratio**: $\Pi_n = \frac{M_{Ln}}{M_{0,1}} = \Pi_{i=1}^{n}\left(\frac{\lambda_i}{1+\lambda_i}\right)$

Note that only two of the three parameters are independent. Since the final velocity of the i^{th} stage is the initial velocity of the $(i+1)^{th}$ stage, the final velocity increment after n stages reads

$$\begin{aligned}\Delta v|_{space} &= v_{f,n} - v_{0,1} \\ &= (v_{f,n} - v_{0,n}) + (v_{f,n-1} - v_{0,n-1}) + ... + (v_{f,1} - v_{0,1}) \\ &= \sum_{i=1}^{n} \Delta v_i\end{aligned} \tag{5.14}$$

which can be written as

$$\begin{aligned}\Delta v|_{space} &= \sum_{i=1}^{N} V_{ei} \ln R_i \\ &= \sum_{i=1}^{N} V_{ei} \ln \frac{1+\lambda_i}{\epsilon_i + \lambda_i}\end{aligned} \tag{5.15}$$

where V_{ei} is the exhaust velocity at the i^{th} stage.

6 Optimizing Multi-stages Rocket

The effective exhaust velocities , V_{ei} and the structural coefficient ,ϵ_i, are known constants based on some prior choice of propellants and structural design for each stage. The question is given $\Delta v|_{space}$, how should one distribute the total mass of the vehicle among the various stages so as to maximize the payload fraction[1]. So, the problem is to maximize

$$\lambda_* = \Pi_{i=1}^{n} \frac{\lambda_i}{1+\lambda_i} \tag{6.1}$$

with the constraint (5.15). In this case one introduce a Lagrange multiplier γ, such that Maximizing $\ln \Gamma$ is equivalent to maximizing

$$\frac{\partial \lambda_*}{\partial \lambda_i} + \gamma \frac{\partial \left(\Delta v|_{space}\right)}{\partial \lambda_i} = 0; \quad i = 1, 2, .., n \tag{6.2}$$

[1]One also could ask the question: given Π what is the maximum final velocity.

which yields

$$\frac{\lambda_*}{\lambda_i\left(1+\lambda_i\right)}-\gamma V_{ei}\frac{1-\epsilon_i}{1+\lambda_i}\frac{1}{\epsilon_i+\lambda_i}=0;\quad i=1,2,..,n \tag{6.3}$$

Similarity of weight ratios for optimized rockets implies that

$$V_{ei}\lambda_i\frac{1-\epsilon_i}{\epsilon_i+\lambda_i}=\frac{\lambda_*}{\gamma}=:\alpha=Const \tag{6.4}$$

Solving for λ_i gives

$$\lambda_i^{(Optimal)}=\frac{\alpha\epsilon}{\left(1-\epsilon_i\right)V_{ei}-\alpha} \tag{6.5}$$

which when inserted in (6.6) and (5.15), yields

$$\lambda_*=\Pi_{i=1}^{n}\frac{\alpha\epsilon_i}{\left(1-\epsilon_i\right)\left(V_{ei}-\alpha\right)} \tag{6.6}$$

$$\Delta v|_{space}=\sum_{i=1}^{n}V_{ei}\ln\left(\frac{V_{ei}-\alpha}{\epsilon_i V_{ei}}\right) \tag{6.7}$$

Fo a given $\Delta v|_{space}$, the constant α is numerically calculated from (6.7) and inserted in (6.6) to determine the maximized total payload λ_*.

Let us assume that the exhaust velocity and the structural coefficients is the same for all stages, i.e. $V_{ei}=V_e$ and $\epsilon_i=\epsilon$. If we define

$$\beta_n=\frac{\Delta v|_{space}}{nV_e} \tag{6.8}$$

then, we have

$$\alpha=V_e\left[1-\epsilon e^{\beta_n}\right] \tag{6.9}$$

$$\lambda=\frac{1-\epsilon e^{\beta_n}}{e^{\beta_n}-1} \tag{6.10}$$

$$\Pi_n=\left[\frac{e^{-\beta_n}-\epsilon}{\left(1-\epsilon\right)}\right]^n \tag{6.11}$$

$$R=e^{\beta_n} \tag{6.12}$$

Note that for very large n, i.e. $n\to\infty$, the total payload fraction becomes

$$\lim_{n\to+\infty}\Pi_n=\exp\left[-\frac{\Delta v|_{space}}{V_e\left(1-\epsilon\right)}\right] \tag{6.13}$$

In figure.2, we plot the total payload ratio Π_n versus $\Delta v|_{space}/V_e$ for $\epsilon=0.075$ and different values of the number of stages.

REMARQUES

- (ϵ, π) **parameterization of the staging**

  It is worth mentioning that some authors instead of choosing (λ_i, ϵ_i) as parameters for the staging of the rocket, they use (π_i, ϵ_i), where

$$\pi_i = \frac{M_{0,i+1}}{M_{0,i}} \tag{6.14}$$

  is the payload fraction at the i^{th} stage. Then the final velocity increment reads

$$\Delta v|_{space} = -\sum_{i=1}^{n} V_{ei} \ln\left[\epsilon_i + (1-\epsilon_i)\,\pi_i\right] \tag{6.15}$$

  If as before, we assume that $V_{ei} = V_e$ and $\epsilon_i = \epsilon$, then to find the optimum values of payload, one must maximize

$$\ln \Gamma = \sum_{i=1}^{n} \ln \pi_i + \Lambda \left[\frac{\Delta v|_{space}}{n} + V_e \ln\left[\epsilon + (1-\epsilon)\,\pi_i\right]\right] \tag{6.16}$$

  where we introduced the Lagrange multiplier Λ to take into account the constraint (6.15). By taking the partial derivative w.r.t π_i and set equal to zero, we get

$$\frac{1}{\pi_i} + \Lambda \frac{V_e\,(1-\epsilon)}{\epsilon + (1-\epsilon)\,\pi_i} = 0 \tag{6.17}$$

  Solving for the payload ratios we obtain

$$\pi_i = -\frac{\epsilon}{(1-\epsilon)\,(1+\Lambda V_e)} \tag{6.18}$$

  which is the same for each stage. Substituting the above expression of π_i in (6.15) gives

$$\Delta v|_{space} = -\sum_{i=1}^{n} V_e \ln\left[\epsilon + \frac{\epsilon}{1+\Lambda V_e}\right] \tag{6.19}$$

  from which we can get

$$\Lambda = \frac{e^{-\beta_n} - 2\epsilon}{V_e\left(\epsilon - e^{-\beta_n}\right)} \tag{6.20}$$

  Using (6.18), we find that optimal value of payload is

$$\pi_i = \frac{e^{-\beta_n} - \epsilon}{(1-\epsilon)} \tag{6.21}$$

  Thus, for n identical stages, the total payload ratio is

$$\Pi_n = \left[\frac{e^{-\beta_n} - \epsilon}{(1-\epsilon)}\right]^n \tag{6.22}$$

  which is the same expression as in (6.11).

Examples

- Consider a liquid oxygen, kerosene system, and take
  - The specific impulse $I_{sp} = 360\ s$, which implies that $V_e = 3528\ m/s$.
  - The final speed increment $\Delta v|_{space} = 9077\ m/s$, needed to reach the orbital speed.
  - The structural coefficient $\epsilon = 0.1$.
  - The number of stages $n = 3$

  With these numbers, the design results are

$$\begin{aligned} \alpha &= 2696\ m/s \\ \lambda^{(Optimal)} &= 0.5 \\ R &= 2.36 \\ \Gamma^{(Optimal)} &= 0.047 \end{aligned} \tag{6.23}$$

  Thus, less than 5% of the overall mass of the vehicle is payload.

- The Saturn V rocket engines that sent astronauts to the Moon, had three stages:
  (a) First stage: $M_{01} = 2902\ t, \epsilon_1 = 0.0765, V_{e1} = 2.98\ km/s$.

  (b) Second stage: $M_{02} = 658\ t, \epsilon_2 = 0.114, V_{e2} = 4.13\ km/s$.

  (c) Third stage: $M_{03} = 165\ t, \epsilon_3 = 0.111, V_{e3} = 4.13\ km/s$.

  which gives a payload ratio of $\lambda_* = 1.62\%$, i.e. 47 t of payload, at $\Delta v|_{space} = 12.4\ km/s$.

7 Derivation of the Exhaust Velocity of a Rocket

8 Rocket Design Principles

1. Discuss the components of a rocket ect...

2. To maximize $\Delta v|_{space}$, both rocket exhaust speed, V_e (or I_{sp}), and mass ratio, R, must be maximized. The exhaust speed derives from propulsion system performance, while mass ratio is a figure of merit of the structural.

3. Typically, the propulsion $Trust/Weight$ is more significant during early boost (1 st stage) and I_{sp} is important during last boost or for upper stages.

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