Newton's method with sixth-order convergence for solving systems of non-linear equations

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ABSTRACT
We extend to the p-dimensional problems a modified Newton’s method with sixth-order convergence. A general error analysis providing the higher order of convergence is given. This new method may be more efficient then other high-order methods, as this does not require the use of the second-order Frechet derivative. Keywords: Newton’s method, Sixth-order convergence, Error analysis, Evaluation of the inversion of operator.

1 INTRODUCTION
In recent paper [2] a sixth-order convergence method has been obtained, by using Newton’s theorem for the function on a new interval of integration. More precisely Newton’s method may be seen as the approximation of the indefinite integral arising from Newton’s theorem [4]
\[ f(x) = f(x_0) + \int_{x_0}^{x} f'(t) dt \] (1.1)
by using the rectangular rule (the Newton-Cotes quadrature formula of order zero) for the computation of the integral in (1.1)
\[ \int_{x_0}^{x} f'(t) dt \approx (x-x_0) f'(x_0) \]
And, looking for \( f(x) = 0 \), we obtain the new value
\[ x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \] (1.2)
Weerakoom - Fernando [6] used the trapezoidal rule to approximate the right integral of (1.1), and obtain the implicit method
\[ x_{n+1} = x_n - \frac{2}{f(x_n)} f'(x_{n+1}) + f'(x_n) \] (1.3)
and, replacing \( f'(x_{n+1}) \) with \( f'(x_{n+1}^*) \); where
\[ x_{n+1}^* = x_n - f(x_n)/f'(x_n) \] is the classical Newton’s iterate, they obtained the explicit with third-order convergence:
\[ x_{n+1} = x_n - \frac{2}{f(x_n)} f'(x_n) \] (1.4)
\[ x_{n+1}^* = x_n - f(x_n)/f'(x_n) \] (1.5)
The midpoint rule for the integral of (1.1) gives that [4]
\[ x_{n+1} = x_n - \frac{f(x_n)}{f(x_n + x_{n+1}/2)} \] (1.6)

In paper [3] Kou, Li, and Wang presented another modification of Newton’s method, based on modification of Weerakoom - Fernando [6] approach by using Newton’s theorem for the function on a new interval of integration, and obtained the following modified Newton’s method with third-order convergence
\[ f(x) = f(y_n) + \int_{y_n}^{x} f'(t) dt \] (1.7)
where \( y_n = x_n + f(x_n)/f'(x_n) \).
by using the mid-point rule to approximate the right integral of (1.6)
\[ \int_{y_n}^{x} f'(t) dt \approx (x-y_n) f'(x_n) \] (1.8)
A general implicit method is then obtained by approximating the integral in (1.6) by the mid-point rule (1.7), and looking for \( f(x) = 0 \), then
\[ x_{n+1} = y_n - \frac{f(y_n)}{f'(x_n + y_n)} \]
they obtained an explicit method by replacing
\[ f'(x_{n+1} + y_n) \] with \( f'(x_{n+1}^* + y_n) \); where
\[ x_{n+1}^* = x_n - f(x_n)/f'(x_n) \] is the classical Newton’s iterate, so the new method is
\[ y_n = x_n + f(x_n)/f'(x_n) \] (1.9)
\[ x_{n+1} = y_n - \frac{f(y_n)}{f'(x_n)} \]
And, In [2], as modification of Kou, Li, and Wang [3] approach, we considered the computation of the indefinite integral on a new interval of integration arising from Newton’s theorem
\[ f(x) = f(z_{n+1}) + \int_{z_{n+1}}^{x} f'(t) dt \] (1.10)
where \( z_{n+1} \) is the above modifications of Newton’s method.
by using the rectangular rule to approximate the right integral of (1.10), we obtained:
\[ x_{n+1} = z_{n+1} - \frac{f(z_{n+1})}{f'(z_{n+1})} \]  

(1.11)

**2 THE P-DIMENSIONAL CASE**

In order to extend to the \( p \)-dimensional case the method proposed in [2], we make use of the following results.

Let \( F: A \subseteq \mathbb{R}^p \rightarrow \mathbb{R}^p \), a vector of \( p \) functions \( k \)-times Fréchet differentiable in a convex set \( A_0 \subset A \), then for any \( X, X_n \in A_0 \), we may write (see [5]) the Taylor’s expansion for \( F \):

\[ F(X) = F(X_n) + F'(X_n)(X - X_n) + \frac{1}{2} F''(X_n)(X - X_n)^2 + \ldots + \frac{1}{(k-1)!} F^{(k-1)}(X_n)(X - X_n)^{k-1} \]

\[ + \int_0^1 (1-t)^{(k-1)} F^{(k)}(X_n + t(X - X_n))(X - X_n)^k dt. \]  

(2.1)

If \( F(\xi) = 0 \), Newton’s method, to approximate the vector \( \xi \), may be obtained considering (2.1) for \( k=1 \):

\[ F(X) = F(X_n) + \int_0^1 F'(X_n + t(X - X_n))(X - X_n) dt. \]

(2.2)

approximating the integral in (2.2) by

\[ \int_0^1 F'(X_n + t(X - X_n))(X - X_n) dt \approx F'(X_n)(X - X_n) \]

and, looking for \( F(X) = 0 \), we obtain

\[ F'(X_n)(X - X_n) = -F(X_n) \]

so that

\[ X_{n+1} = X_n - F'(X_n)^{-1}F(X_n) \]

In [5], Frontini and Sormani approximated the integral in (2.2), by using a quadrature formula with order higher than zero.

1- **The trapezoidal rule**

\[ \int_0^1 F'(X_n + t(X - X_n))(X - X_n) dt \approx \int_0^1 \frac{1}{2} (F'(X_n) + F'(X_{n+1}))(X - X_n) \]

where \( X_{n+1} = X_n - F'(X_n)^{-1}F(X_n) \).

and, looking for \( F(X) = 0 \), they obtained

\[ \frac{1}{2} (F'(X_n) + F'(X_{n+1}))(X - X_n) = -F(X_n) \]

\[ \Rightarrow \]

\[ X_{n+1} = X_n - \frac{1}{2} (F'(X_n) + F'(X_{n+1}))^{-1}F(X_n). \]

(2.3)

2- **The mid-point rule**

\[ \int_0^1 F'(X_n + t(X - X_n))(X - X_n) dt \approx \int_0^1 \frac{1}{2} (F'(X_{n+1} + X_n)) (X - X_n) \]

where \( X_{n+1} = X_n - F'(X_n)^{-1}F(X_n) \), and, looking for \( F(X) = 0 \), the new value

\[ F\left( \frac{1}{2} (X_{n+1} + X_n) \right) (X - X_n) = -F(X_n) \]

\[ \Rightarrow \]

\[ X_{n+1} = X_n - F\left( \frac{1}{2} (X_{n+1} + X_n) \right)^{-1}F(X_n). \]

(2.4)

Now, for any \( X, Y_n \in A_0 \), and \( k=1 \), we can write the Taylor’s expansion for \( F \) as this:

\[ F(X) = F(Y_n) + \int_0^1 F'(Y_n + t(X - Y_n))(X - Y_n) dt. \]

(2.5)

By using the mid-point rule to approximate the integral in (2.5)

\[ \int_0^1 F(Y_n + t(X - Y_n))(X - Y_n) dt \approx F\left( \frac{1}{2} (Y_n + X_n) \right) (X - Y_n) \]

where \( Y_n = X_n + F'(X_n)^{-1}F(X_n) \), and

\[ X_{n+1} = X_n - F'(X_n)^{-1}F(X_n) \]

and, looking for \( F(X) = 0 \), we obtained the method

\[ X_{n+1} = Y_n - F'(X_n)^{-1}F(Y_n) \]

\[ \Rightarrow \]

\[ X_{n+1} = X_n - F'(X_n)^{-1}\left( F(X_n) + F'(X_n)^{-1}F(X_n) \right) - F(X_n) \]

(2.6)

Again, for any \( X, Z_{n+1} \in A_0 \), and \( k=1 \), we can writing the Taylor’s expansion for \( F \) as this:

\[ F(X) = F(Z_{n+1}) + \int_0^1 F'(Z_{n+1} + t(X - Z_{n+1}))(X - Z_{n+1}) dt. \]

(2.7)

we use the rectangular rule to approximate the right integral of (2.7)

\[ \int_0^1 F'(Z_{n+1} + t(X - Z_{n+1}))(X - Z_{n+1}) dt \approx F'(Z_{n+1}) (X - Z_{n+1}) \]

where \( z_{n+1} \) is the above modifications of Newton’s method (2.3), (2.4), and (2.6).

and, looking for \( F(X) = 0 \), we obtained the new method

\[ X_{n+1} = Z_{n+1} - F'(Z_{n+1})^{-1}F(Z_{n+1}) \]

(2.8)

**Theorem 1**

Let \( F: A \subseteq \mathbb{R}^p \rightarrow \mathbb{R}^p \), three times Fréchet differentiable in a convex set \( A \) containing the root \( \xi \) of \( F(X) = 0 \). The modified Newton’s method, \( mNM \) (2.8) has order of convergence six.

**Proof:**

From Eq.(2.8), defining \( e_n = X_n - \xi \), we have:

\[ X_{n+1} = Z_{n+1} - F'(Z_{n+1})^{-1}F(Z_{n+1}) \]

\[ e_{n+1} = Z_{n+1} - \xi - F'(Z_{n+1})^{-1}F(Z_{n+1}) \]

(2.9)

Form Eq.(2.1), with \( k=3 \), \( X = \xi \), and \( X_n = Z_{n+1} \), we have:
Newton's method with sixth-order convergence for solving systems of non-linear equations

\[ F(\mathbf{\xi}) = F(Z_{n+1}) + F'(Z_{n+1})[\mathbf{\xi} - Z_{n+1}] + \frac{1}{2} F''(Z_{n+1})[\mathbf{\xi} - Z_{n+1}]^2 + O\left(\|\mathbf{\xi} - Z_{n+1}\|^3\right). \]

Being \( F(\mathbf{\xi}) = 0 \) we have:

\[ F(Z_{n+1}) = F'(Z_{n+1})[Z_{n+1} - \mathbf{\xi}] - \frac{1}{2} F''(Z_{n+1})[Z_{n+1} - \mathbf{\xi}]^2 + O\left(\|Z_{n+1} - \mathbf{\xi}\|^3\right) \]

(2.10)

Form Eq.(2.10) we may write the product \( F(Z_{n+1}) \) in Eq.(2.9) as:

\[ e_{n+1} = (Z_{n+1} - \mathbf{\xi}) - F'(Z_{n+1})^{-1}\left[ F'(Z_{n+1})[Z_{n+1} - \mathbf{\xi}] - \frac{1}{2} F''(Z_{n+1})[Z_{n+1} - \mathbf{\xi}]^2 + O\left(\|Z_{n+1} - \mathbf{\xi}\|^3\right) \right] \]

(2.11)

Form Eq.(2.1), with \( k=4 \) and \( X = \mathbf{\xi} \), we have:

\[ F'(X_{n}) = F'(X_{n}) + F'(X_{n})[\mathbf{\xi} - X_{n}] + \frac{1}{2} F''(X_{n})[\mathbf{\xi} - X_{n}]^2 + \frac{1}{3} F'''(X_{n})[\mathbf{\xi} - X_{n}]^3 + o(\|\mathbf{\xi} - X_{n}\|^4) \]

= \( F(X_{n}) - F'(X_{n})e_{n} + \frac{1}{2} F''(X_{n})e_{n}^2 - \frac{1}{3} F'''(X_{n})e_{n}^3 + o(\|e_{n}\|^4) \) .

Being \( F(\mathbf{\xi}) = 0 \) we have:

\[ F(X_{n}) = F'(X_{n})e_{n} - \frac{1}{2} F''(X_{n})e_{n}^2 + \frac{1}{3} F'''(X_{n})e_{n}^3 + o(\|e_{n}\|^4) \]  

(2.12)

\[ F'(X_{n})e_{n} = F'(X_{n}) + \frac{1}{2} F''(X_{n})e_{n}^2 - \frac{1}{3} F'''(X_{n})e_{n}^3 + o(\|e_{n}\|^4) \]  

(2.13)

Form Eq.(2.12) we may write the \( F'(X_{n})^{-1} F(X_{n}) \) as:

\[ F'(X_{n})^{-1} F(X_{n}) = F'(X_{n})^{-1} \left[ F'(X_{n})e_{n} - \frac{1}{2} F''(X_{n})e_{n}^2 + \frac{1}{3} F'''(X_{n})e_{n}^3 + o(\|e_{n}\|^4) \right] \]

\[ = e_{n} - \frac{1}{2} F'(X_{n})^{-1} F'(X_{n})e_{n}^2 + \frac{1}{3} F'(X_{n})^{-1} F''(X_{n})e_{n}^3 + o(\|e_{n}\|^4) \]  

(2.14)

by considering Eq.(2.1), we have the Taylor’s expansion of \( F(X_{n}) + F'(X_{n})^{-1} F(X_{n}) \) as:

\[ F'(X_{n}) - F'(X_{n})^{-1} F(X_{n}) = F'(X_{n}) - F'(X_{n})F'(X_{n})^{-1} F(X_{n}) + \frac{1}{2} F'(X_{n}) F'(X_{n})^{-1} F'(X_{n})^{-1} F(X_{n}) \]

\[ + \frac{1}{2} F''(X_{n}) F'(X_{n})^{-1} F(X_{n})^2 + O(\|e_{n}\|^3) \]

using Eq.(2.14) we obtain:

\[ F'(X_{n}) - F'(X_{n})^{-1} F(X_{n}) = \left[ e_{n} - \frac{1}{2} F'(X_{n})^{-1} F'(X_{n})e_{n}^2 + \frac{1}{3} F'(X_{n})^{-1} F''(X_{n})e_{n}^3 + o(\|e_{n}\|^4) \right] \]

\[ + \frac{1}{2} F''(X_{n}) e_{n}^2 + \frac{1}{3} F''(X_{n}) e_{n}^3 + o(\|e_{n}\|^4) \]

\[ = \frac{1}{2} F''(X_{n}) e_{n} + \frac{1}{8} F''(X_{n}) e_{n}^3 + o(\|e_{n}\|^4) \]

\[ \Rightarrow F'(X_{n}) - F'(X_{n})^{-1} F(X_{n}) = F'(X_{n}) e_{n} + \frac{1}{2} F''(X_{n}) e_{n}^3 + \frac{1}{8} F''(X_{n}) e_{n}^3 + o(\|e_{n}\|^4) \]

by considering Eq.(2.1) we have the Taylor’s expansion of \( F(X_{n}) + F'(X_{n})^{-1} F(X_{n}) \) as:

\[ F(X_{n}) + F'(X_{n})^{-1} F(X_{n}) = F(X_{n}) + F'(X_{n}) F'(X_{n})^{-1} F(X_{n}) + \frac{1}{2} F''(X_{n}) F'(X_{n})^{-1} F'(X_{n})^{-1} F(X_{n}) \]

\[ + \frac{1}{2} F''(X_{n}) F'(X_{n})^{-1} F'(X_{n})^{-1} F'(X_{n})^{-1} F(X_{n}) \]

\[ + \frac{1}{2} F''(X_{n}) F'(X_{n})^{-1} F'(X_{n})^{-1} F'(X_{n})^{-1} F'(X_{n})^{-1} F(X_{n}) \]

\[ + \frac{1}{4} F''(X_{n}) F'(X_{n})^{-1} F'(X_{n})^{-1} F'(X_{n})^{-1} F'(X_{n})^{-1} F'(X_{n})^{-1} F(X_{n}) \]

\[ + \frac{1}{8} F''(X_{n}) F'(X_{n})^{-1} F'(X_{n})^{-1} F'(X_{n})^{-1} F'(X_{n})^{-1} F'(X_{n})^{-1} F'(X_{n})^{-1} F(X_{n}) \]

\[ + O(\|e_{n}\|^4) \]

using Eq.(2.14) we obtain:
\[ F(X_n + F'(X_n))^{-1} F(X_n) = \]
\[ 2F(X_n) + \frac{1}{2} F'(X_n) \left[ e_n - \frac{1}{2} F'(X_n) F'(X_n) e_n^2 + \frac{1}{3} F'(X_n) F'(X_n) e_n^3 \right] \]
\[ + \frac{1}{3!} F'(X_n) F'(X_n) F'(X_n) e_n^3 + \frac{1}{2} F'(X_n) F'(X_n) e_n^3 + \frac{1}{3} F'(X_n) F'(X_n) e_n^3 + \frac{1}{6} F'(X_n) F'(X_n) e_n^3 + \frac{1}{24} F'(X_n) F'(X_n) e_n^3 \]
\[ + \frac{1}{24} F'(X_n) F'(X_n) e_n^3 + \frac{1}{2} F'(X_n) F'(X_n) e_n^3 + \frac{1}{3} F'(X_n) F'(X_n) e_n^3 + \frac{1}{6} F'(X_n) F'(X_n) e_n^3 + \frac{1}{24} F'(X_n) F'(X_n) e_n^3 + \frac{1}{24} F'(X_n) F'(X_n) e_n^3 + \frac{1}{24} F'(X_n) F'(X_n) e_n^3 + \frac{1}{24} F'(X_n) F'(X_n) e_n^3 \]
\[ + \text{higher terms} \]

2- For \( Z_{n+1} \) defined by formula (2.4)
\[ Z_{n+1} = X_n - F'(X_n)^{-1} F(X_n) \]
\[ \Rightarrow \]
\[ Z_{n+1} - \xi = (X_n - \xi) - F'(X_n)^{-1} F(X_n) - F'(X_n)^{-1} F(X_n) \]
So that
\[ F(X_n) - F'(X_n)^{-1} F(X_n) (Z_{n+1} - \xi) = \]
\[ \left( F'(X_n) F'(X_n) e_n^3 + \frac{1}{2} F'(X_n) F'(X_n) e_n^3 + \frac{1}{3} F'(X_n) F'(X_n) e_n^3 + \frac{1}{6} F'(X_n) F'(X_n) e_n^3 + \frac{1}{24} F'(X_n) F'(X_n) e_n^3 + \frac{1}{24} F'(X_n) F'(X_n) e_n^3 + \frac{1}{24} F'(X_n) F'(X_n) e_n^3 + \frac{1}{24} F'(X_n) F'(X_n) e_n^3 \right) \]
\[ = 2 F(X_n) + \frac{1}{2} F'(X_n) e_n^2 + \frac{1}{3} F'(X_n) e_n^3 + \frac{1}{6} F'(X_n) e_n^3 + \frac{1}{24} F'(X_n) e_n^3 \]
\[ + \frac{1}{24} F'(X_n) e_n^3 + \frac{1}{24} F'(X_n) e_n^3 + \frac{1}{24} F'(X_n) e_n^3 + \frac{1}{24} F'(X_n) e_n^3 \]
\[ + \frac{1}{24} F'(X_n) e_n^3 + \frac{1}{24} F'(X_n) e_n^3 + \frac{1}{24} F'(X_n) e_n^3 + \frac{1}{24} F'(X_n) e_n^3 \]
\[ + \frac{1}{24} F'(X_n) e_n^3 \]

1- For \( Z_{n+1} \) defined by formula (2.3)
2-\]
\[ Z_{n+1} = X_n - 2 F'(X_n - F'(X_n)^{-1} F(X_n) + F'(X_n)^{-1} F(X_n) \]
\[ \Rightarrow \]
\[ Z_{n+1} - \xi = (X_n - \xi) - 2 F'(X_n - F'(X_n)^{-1} F(X_n) + F'(X_n)^{-1} F(X_n) \]
So that
\[ \left( F'(X_n - F'(X_n)^{-1} F(X_n) + F'(X_n)^{-1} F(X_n) \right) Z_{n+1} - \xi = \]
\[ \left( F'(X_n - F'(X_n)^{-1} F(X_n) + F'(X_n)^{-1} F(X_n) \right) e_n - 2 F(X_n) \]
form (2.12) and (2.15) we obtain :
\[ \left( F'(X_n - F'(X_n)^{-1} F(X_n) + F'(X_n)^{-1} F(X_n) \right) \left( e_n - 2 F(X_n) \right) \]
\[ = 2 F(X_n) + \frac{1}{2} F'(X_n) e_n^2 + \frac{1}{3} F'(X_n) e_n^3 + \frac{1}{6} F'(X_n) e_n^3 + \frac{1}{24} F'(X_n) e_n^3 \]
\[ + \frac{1}{24} F'(X_n) e_n^3 + \frac{1}{24} F'(X_n) e_n^3 + \frac{1}{24} F'(X_n) e_n^3 + \frac{1}{24} F'(X_n) e_n^3 \]
\[ + \frac{1}{24} F'(X_n) e_n^3 \]
\[ = \left( F'(X_n - F'(X_n)^{-1} F(X_n) + F'(X_n)^{-1} F(X_n) \right) e_n^3 + O(||e_n||^4) \]
\[ = \]
\[ \left( \left( F'(X_n - F'(X_n)^{-1} F(X_n) + F'(X_n)^{-1} F(X_n) \right) e_n^3 + O(||e_n||^4) \right) \]
\[ \Rightarrow \]
\[ Z_{n+1} - \xi = \]
\[ \left( \left( F'(X_n - F'(X_n)^{-1} F(X_n) + F'(X_n)^{-1} F(X_n) \right) e_n^3 + O(||e_n||^4) \right) \]
\[ = \]
\[ \left( \left( F'(X_n - F'(X_n)^{-1} F(X_n) + F'(X_n)^{-1} F(X_n) \right) e_n^3 + O(||e_n||^4) \right) \]
\[ = \]
\[ \left( \left( F'(X_n - F'(X_n)^{-1} F(X_n) + F'(X_n)^{-1} F(X_n) \right) e_n^3 + O(||e_n||^4) \right) \]
\[ = \]
\[ \left( \left( F'(X_n - F'(X_n)^{-1} F(X_n) + F'(X_n)^{-1} F(X_n) \right) e_n^3 + O(||e_n||^4) \right) \]
\[ = \]
\[ \left( \left( F'(X_n - F'(X_n)^{-1} F(X_n) + F'(X_n)^{-1} F(X_n) \right) e_n^3 + O(||e_n||^4) \right) \]
\[ = \]
\[ \left( \left( F'(X_n - F'(X_n)^{-1} F(X_n) + F'(X_n)^{-1} F(X_n) \right) e_n^3 + O(||e_n||^4) \right) \]
\[ = \]
\[ \left( \left( F'(X_n - F'(X_n)^{-1} F(X_n) + F'(X_n)^{-1} F(X_n) \right) e_n^3 + O(||e_n||^4) \right) \]
\[ = \]
\[ \left( \left( F'(X_n - F'(X_n)^{-1} F(X_n) + F'(X_n)^{-1} F(X_n) \right) e_n^3 + O(||e_n||^4) \right) \]
\[ = \]
\[ \left( \left( F'(X_n - F'(X_n)^{-1} F(X_n) + F'(X_n)^{-1} F(X_n) \right) e_n^3 + O(||e_n||^4) \right) \]
\[ = \]
\[ \left( \left( F'(X_n - F'(X_n)^{-1} F(X_n) + F'(X_n)^{-1} F(X_n) \right) e_n^3 + O(||e_n||^4) \right) \]
\[ = \]
\[ \left( \left( F'(X_n - F'(X_n)^{-1} F(X_n) + F'(X_n)^{-1} F(X_n) \right) e_n^3 + O(||e_n||^4) \right) \]
\[ = \]
\[ \left( \left( F'(X_n - F'(X_n)^{-1} F(X_n) + F'(X_n)^{-1} F(X_n) \right) e_n^3 + O(||e_n||^4) \right) \]
Newton’s method with sixth-order convergence for solving systems of non-linear equations

\[ F'(X_n)(Z_{n+1} - \xi) = \]
\[ \left[ F(X_n) + \frac{1}{2} F'(X_n) e_n^2 - \frac{1}{3!} F''(X_n) e_n^3 + o(\|e_n\|^3) \right] - \]
\[ \left\{ F(X_n) + \frac{1}{2} F'(X_n) e_n^2 - \frac{1}{2} F''(X_n) F'(X_n)^{-1} F'(X_n) e_n^3 + \frac{1}{3!} F''(X_n) e_n^3 + o(\|e_n\|^3) \right\} \]
\[ = \frac{1}{2} F''(X_n) [F'(X_n)]^{-1} F'(X_n) e_n^3 + o(\|e_n\|^3). \]
\[ Z_{n+1} - \xi = F'(X_n)^{-1} \left[ \frac{1}{2} F''(X_n) F'(X_n)^{-1} F'(X_n) e_n^3 + \frac{1}{2} F''(X_n) e_n^3 + o(\|e_n\|^3) \right] \]
\[ + o(\|e_n\|^3) \] (2.20)

form (2.11) and (2.20) we obtain:
\[ e_{n+1} = F'(Z_{n+1})^{-1} \frac{1}{2} F'(Z_{n+1}) \times \]
\[ \left\{ F'(X_n)^{-1} \left[ \frac{1}{2} F''(X_n) F'(X_n)^{-1} F'(X_n) e_n^3 \right] \right\}^2 \]
\[ e_n^6 + O(\|e_n\|^7). \]

3 CONCLUSIONS

Without going into computational details, we point out that the method (2.8) requires two evaluations of the inversion of operator in per iteration when \( Z_{n+1} \) defined by formula (2.6) , like modified Newton’s methods (2.3) and (2.4) that have convergence of third-order, but it has higher order convergence, while the method (2.8) requires three evaluations of the inversion of operator in per iteration when \( Z_{n+1} \) defined by formula (2.3) or formula (2.4).

The classical Newton’s method also requires only evaluation of the inversion of operator, but it’s a second-order convergence. However, in some applications that require a number of repeated tests, the increasing of order may draw a concrete advantage if the cost of computational complexity, implied by the increased order, is justified from the advantage in terms of execution time.

REFERENCES

H. M. Eldarfour, Modified Newton’s methods with fifth or sixth –order convergence.