A comparison between Runge-Kutta method and optimization method for nonlinear systems of ordinary differential equations

Ali Asghar Behroozpoor¹, Mohammad Mehdi Mazarei²
¹Department of Science, Islamic Azad University, Bushehr Branch, Bushehr, Iran, behroozpoor@iaabushehr.ac.ir, Funded by the office of vice chancellor for research, Islamic Azad University, Bushehr Branch.
²Department of Science, Islamic Azad University, Bushehr Branch, Bushehr, Iran, mazarei@iaabushehr.ac.ir, Funded by the office of vice chancellor for research, Islamic Azad University, Bushehr Branch.

ABSTRACT
This study presents a typical optimization approach for numerical solution of nonlinear systems of ordinary differential equations (ODEs). We make a comparison between Runge-Kutta method and optimization method for nonlinear these. These nonlinear systems are due to many fields such as biology, control systems, momentum, energy and species transfer problems. In this work, we have solved nonlinear systems of ODEs by transmitting them to an optimum problem that its target function is the norm of error. Then, we try to decrease norm of error as much as possible. This technique achieves better solutions in comparison with Runge-Kutta method.

ICM 2012. Keywords: Numerical solution; Optimum problem; Nonlinear systems; Ordinary differential equations.

1 INTRODUCTION
The non linear systems of differential equations always have been a challengeable object in applied mathematics. Analytical methods are frequently inadequate for obtaining solution and numerical methods must be resorted to. Also, traditional numerical methods such as finite element method, finite difference method, Runge-Kutta method and so on are faced to some difficulties. So, many researchers have been looking for new approaches to obtain more accurate numerical solutions of differential equations.

One of the first researchers that has applied such this technique was Gear and et al [1]. They use it for numerical solution of ordinary differential equations. Also, Ranji Srinivasan and et al [2] used this method to model electrical simulate of cardiac heat cells. Robin Raffard and et al [3] have used this approach to solve partial differential equations that have occurred in air traffic flow. Recently, Ali Asghar Behroozpoor and Mohammad Mehdi Mazarei [4] have used this method to simulate heat cell electrical activity. In this work, we have used this technique for numerical solution of nonlinear systems of ordinary differential equations.

2 THE GENERAL FORM OF A NONLINEAR SYSTEM
The general form of a linear time independent control system is as below:

\[ x'(t) = Ax(t) + Bu(t) \] (1)

where \( A \in \mathbb{R}^{n \times n} \) and \( B \in \mathbb{R}^{n \times m} \) are two constant matrices and \( x(t) \), \( u(t) \) are two vectors in spaces \( \mathbb{R}^n \) and \( \mathbb{R}^m \), respectively. Also, the general form of a linear time dependent control system is as below:

\[ x'(t) = A(t) x(t) + B(t) u(t) \] (2)

A nonlinear control system is in the form

\[ \dot{x}(t) = g(x(t),u(t),t) \] (3)

Where \( g \) is a nonlinear function of \( x(t) \) and \( u(t) \). We consider this nonlinear system with below conditions:

\[ x(a)=c \quad x(b)=d \] (4)

Where \( c \) and \( d \) are initial and final conditions, respectively that can be constant or variable and \( g \) is a continuous function on \( A \times B \times [a,b] \) such that \( t \in [a,b] \), \( x(t) \in A \subset \mathbb{R}^n \), \( u(t) \in B \subset \mathbb{R}^m \), and \( x(t), u(t) \) must be selected such that our system can be transmitted from \( x(a) \) to \( x(b) \).

3 METHODS
We introduce error function by this relation

\[ E(x'(t),x(t),u(t),t) = \int_a^b \| x'(t) - g(x(t),u(t),t) \| dt \] (5)

On the other hand, we have these important theorems:

**Theorem 1.** If \( h(x) \) is a nonlinear continuous function on \( A \times B \times [a,b] \) and \( 0 \leq h(x) \), then

\[ \iff \quad h(x)=0 \] (6)

\[ \int_a^b h(x) \, dx = 0 \] \quad **Theorem 2.** The necessary and sufficient condition for the nonlinear system (2) to be transmitted from initial condition \( x(a)=c \) to final condition \( x(b)=d \) is that:

\[ E(x'(t),x(t),u(t),t) = 0 \] (7)
Without any damage to the problem’s totality, we can consider  
\( a = 0 \) and  \( b = 1 \). Now, we have the interval  \((a,b)\) is replaced by  
\((0,1)\).
Now, we have this below optimum problem:

\[
\begin{align*}
\text{Min} & \quad E(x'(t),x(t),u(t),t) \\
= & \text{Min} \int_0^b \|x'(t) - g(x(t),u(t),t)\| \, dt
\end{align*}
\]

Such that  
\[
x(0) = x_0 \quad x(1) = x_1
\]

If we consider  \( x^* (t) \), \( u^* (t) \) as the optimized solution of this problem, then

\[
E \left( x^* (t), x^* (t), u^* (t), t \right) = 0
\]

According Theorem 1, we have

\[
x^* (t) = g \left( x^* (t), u^* (t), t \right) \quad x^* (0) = x_0 \quad x^* (1) = x_1
\]

If this problem does not have solution (zero solution), the nearest approximate solution will be obtained.
If we divide interval  \((0,1)\) to  \( n \) subintervals, then function (8) is reduced to below function:

\[
\int_0^b \|x'(t) - g(x(t),u(t),t)\| \, dt = \sum_{i=1}^{n} \int_{(i-1)/n}^{i/n} \|x'(t) - g(x(t),u(t),t)\| \, dt
\]

If  \( \Delta t = 1/n \), with derive approximation

\[
x'(t) \simeq \frac{x(t+\Delta t) - x(t)}{\Delta t}
\]

We know this fraction approaches to derive value on time  \( t \), as  \( \Delta t \) approaches to zero. So, we have to increase  \( n \).
Also, we use the norm  \( L_1' \). Hence, we have relation (11) as below relation:

\[
\int_0^b \|x'(t) - g(x(t),u(t),t)\| \, dt = \sum_{i=1}^{n} \int_{(i-1)/n}^{i/n} \|x'(t) - g(x(t),u(t),t)\| \, dt
\]

and according mean value theorem for integrals, we have

\[
\int_a^b f(x) \, dx = (b-a) f(c)
\]

Where  \( c \) is belong to  \((a,b)\). So, by using relation (14) problem (11) is reduced to this problem:

\[
\begin{align*}
\int_0^b & \|x'(t) - g(x(t),u(t),t)\| \, dt = \sum_{i=0}^{n-1} \frac{1}{n} \left[ \|x(i/n) - x((i-1)/n)\| - \|g(x(i/n),u(i/n),t_i)\| \right] \\
\text{Unknowns are defined as this:} & \\
x_i = \frac{x(i/n)}{n} \quad x_{i+1} = \frac{x((i+1)/n)}{n} \quad u_i = \frac{u(i/n)}{n} \quad t_i = \frac{t_i}{n} \\
t = 0,1,2,\ldots,n
\end{align*}
\]

So, problem (11) could be written as below:

\[
\int_0^b \|x'(t) - g(x(t),u(t),t)\| \, dt = \sum_{i=0}^{n-1} \frac{1}{n} \left[ \|x_{i+1} - x_i\| - \|g(x_i,u_i,t_i)\| \right]
\]

and

\[
x(0) = x_0 \quad x(1) = x_n
\]

Also, we define

\[
r_i - s_i = n(x_{i+1} - x_i) - g(x_i,u_i,t_i) \quad 0 \leq r_i \leq s_i \quad i = 0,1,2,\ldots,n-1
\]

Finally, we have this nonlinear problem

\[
\begin{align*}
\text{Min} & \quad \sum_{i=0}^{n-1} \frac{1}{n} [r_i + s_i] \\
\text{Such that} & \\
r_i - s_i = n(x_{i+1} - x_i) - g(x_i,u_i,t_i) \quad i = 0,1,2,\ldots,n-1 \\
x(0) = x_0 \\
x(1) = x_n \\
0 \leq r_i \leq s_i \\
it = 0,1,2,\ldots,n-1
\end{align*}
\]

4 RESULTS

Experiment1. Consider the following nonlinear system:

\[
\begin{align*}
y'_1 & = t \sin(y_1(t)y_2(t)) + e^{-t^2+y_1(t)}y_1(t)y_2(t) \\
y'_2 & = y_1(t)y_2^2(t) + e^{-y_1(t)y_2(t)}t \\
y_1(0) & = 0.1 \\
y_2(0) & = 0.1 \\
it & \in [0,1]
\end{align*}
\]
As it was explained above, we transmit this nonlinear system of ODEs and its initial conditions to an optimum problem as below:

\[
\text{Min} \quad \frac{1}{N} \sum_{i=0}^{N-1} \sqrt{(r_1(i) - s_1(i))^2 + (r_2(i) - s_2(i))^2}
\]

such that

\[
r_1(i) - s_1(i) = \frac{y_1(t_i + \Delta t) - y_1(t_i)}{\Delta t} - e^{-\gamma_1(t_i)} y_1(t_i) y_2(t_i)
\]

\[
r_2(i) - s_2(i) = \frac{y_2(t_i + \Delta t) - y_2(t_i)}{\Delta t} - e^{-\gamma_2(t_i)} y_1(t_i) y_2(t_i)
\]

\[
r_1(i) \times s_1(i) = 0, \quad i=0,1,\ldots,N-1
\]

\[
r_2(i) \times s_2(i) = 0, \quad i=0,1,\ldots,N-1
\]

\[
y_1(0) = y_2(0) = 0.1
\]

\[
r_1(i), r_2(i) \geq 0, \quad s_1(i), s_2(i) \geq 0
\] (21)

We have solved this optimum problem by software Lingo for some different values of N. In other hand, we have solved this problem by Runge-Kutta method by software MATLAB. We see that this new method gives us more accurate results in comparison Runge-Kutta method (see Table 1).

### Table 1. Comparison between new method and Runge-Kutta method (Experiment 1).

<table>
<thead>
<tr>
<th>N</th>
<th>New method</th>
<th>Runge-Kutta</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>N=10</td>
<td>5.69x10^{-11}</td>
<td>5.06x10^{-3}</td>
</tr>
<tr>
<td>N=20</td>
<td>4.68x10^{-12}</td>
<td>2.67x10^{-3}</td>
</tr>
</tbody>
</table>

### Experiment 2. Consider the following nonlinear system:

\[
y_1' = t \sin(y_1 y_2) + e^{-t y_2} y_1
\]

\[
y_2' = y_1 y_2 y_3^2 + \cos(t y_2^2)
\]

\[
y_3' = e^{-t y_2} + \sin(\tan(y_1 y_2 y_3))
\]

\[
y_1(0) = y_2(0) = y_3(0) = 0.1
\]

\[
t \in [0,1]
\] (22)

Same as experiment 1, we transmit this nonlinear system and its initial conditions to an optimum problem. We have solved this optimum problem by new method for different values N. As earlier experiment, this PDE has been solved by Runge-Kutta method, too. We have find that this new method achieves better solutions than Runge-Kutta method (see Table 2).

### Table 2. Comparison between new method and Runge-Kutta method (Experiment 2).

<table>
<thead>
<tr>
<th>N</th>
<th>New method</th>
<th>Runge-Kutta</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>N=10</td>
<td>6.27x10^{-9}</td>
<td>1.17x10^{-8}</td>
</tr>
<tr>
<td>N=20</td>
<td>7.05x10^{-10}</td>
<td>4.04x10^{-9}</td>
</tr>
</tbody>
</table>

### 5 CONCLUSION

As we can see this new method is more accurate than Runge-Kutta method for nonlinear systems of ordinary differential equations.

### ACKNOWLEDGEMENTS

This research paper has been financially supported by the office of chancellor for research of Islamic Azad University, Bushehr Branch.

### REFERENCES


Robin Raiffard and Claire Tomlin, Second order adjoint-based Optimization of ordinary and partial differential equations with applications to air traffic flow.