Stochastic optimization models in discrete time

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Abstract
In this paper we consider stochastic models of discrete time analogs of optimization models of the Bolza type. The functionals are assumed to be Locally Lipschitzians, but we make no differentiability and convexity assumptions. We have studied in another paper the deterministic models such this theory serves to set the stage for the stochastic model. We obtain optimality conditions that are always sufficient and which are also necessary if the given problem satisfies a qualification condition. Keywords: Nonsmooth analysis, subdifferential, qualification condition, discrete time inclusion, normal integrand, nonanticipativity, conditional expectation.

1 Introduction
Our aim is to treat stochastic versions of discrete time analogs of optimization models of the Bolza type. For this reason we recall the notion of Bolza type model in the classical calculus of variations.

So a model of Bolza type is one where a functional of the form

$$I(x) = l(x(t_0), x(t_1)) + \int_{t_0}^{t_1} L(t, x(t), \dot{x}(t)) \, dt$$

is minimized over a space of arcs $x : [t_0, t_1] \to \mathbb{R}^n$ subject to a system of equations and inequations on the endpoint pair $(x(t_0), x(t_1))$ and the triple $(t, x(t), \dot{x}(t))$. This fundamental dynamical model has in recent years been a focus of efforts towards developing a variational theory not so dependent on smoothness assumptions. In this theory, the constraints are represented by allowing $l$ and $L(t,.,.)$ to be extended real valued functions on $\mathbb{R}^n \times \mathbb{R}^n$, and optimality conditions are expressed in terms of subdifferential: [14].

After taking care of the deterministic case, which is mainly a matter of applying well-known results in nonsmooth sequential analysis and in convex analysis to a particular situation, we study the stochastic version of this class of optimization problems. The significant new feature, not present in the functional form (1.1), is a process that models the flow of information. Decision taken at any time $t$ can only depend on the information collected about past random events, the future being known only in a probabilistic sense. Where as in the deterministic model the decision maker has at any time total information about past and future costs associated with any plan, in the stochastic model at any time $t$, the uncertainly about the actual cost of any decision plan can only be mitigated by past observations see [19, 20].

The stochastic version of our model requires an underlying probability space $(\Omega, \mathcal{A}, \mu)$ and a nest $\zeta$ of $\sigma$-fields: $\zeta = \zeta_0, \zeta_1, \cdots, \zeta_T$, where

$$\zeta_0 \subset \zeta_1 \subset \cdots \subset \zeta_T \subset \mathcal{A}. $$

(2)
The field \( \zeta_t \) represents information available at time \( t \), and to say that a mapping \( x_t : \Omega \to \mathbb{R}^n \) is \( \zeta_t \)-measurable is to say that \( x_t(\omega) \) can depend on such information only, not on unobserved details of past events, or on random events still in the future. Accordingly we restrict attention in our decision-making process to the (closed) linear function space of \( \zeta_t \)-measurable \( x \) of \( L^\infty(\omega, \mathcal{A}, \mu; (\mathbb{R}^n)^{T+1}) \) \( x_t \) is \( \zeta_t \)-measurable. \( x \) is \( \zeta_t \)-measurable and \( x_{t-1} \) is \( \zeta_{t-1} \)-measurable.

As in the deterministic case see [22, 23], \( l \) and the functions \( L_t(\omega, \cdot, \cdot) \), for each \( t = 1, \cdots, T \) and \( \omega \in \Omega \), are locally Lipschitzian on \( \Omega \times \mathbb{R}^n \times \mathbb{R}^n \). Among other things, this ensures that whenever \( z_t(\omega) \) and \( w_t(\omega) \) are \( \zeta_t \)-measurable \( \omega \), so is \( L_t(\omega, z_t(\omega), w_t(\omega)) \). Then, certainly, the term \( L_t(w, x_{t-1}(\omega), \Delta x_t(\omega)) \) is \( \zeta_t \)-measurable for any \( x \in \mathcal{N} \). Thus \( J \) is a well-defined functional on \( \mathcal{N} \) with values in \( \mathbb{R} \cup \{+\infty\} \). In fact \( J \) is locally Lipschitzian ( with respect to the \( L^\infty \)-norm topology on \( \mathcal{N} \)).

Certain constraints are implicit in the stochastic model, just as in the deterministic model, because only the elements \( x \) of \( \mathcal{N} \) which satisfy \( J(x) < \infty \) can be candidates for the minimum of \( J \). Let

\[
F_t(\omega, z_t) := \{w_t \in \mathbb{R}^n \mid L_t(\omega, z_t, w_t) < \infty\}
\]

and

\[
Z_t(\omega) := \{z_t \in \mathbb{R}^n \mid F_t(\omega, z_t) \neq \emptyset\}.
\]

Every \( x \in \mathcal{N} \) must satisfy

\[
(EX_0(\omega), EX_T(\omega)) \in C \text{ and } \Delta x_t(\omega) \in F_t(\omega, x_{t-1}(\omega)) \text{ a.s. for } t = 1, \cdots, T
\]

and consequently \( x_{t-1}(\omega) \in Z_t(\omega) \text{ a.s. for } t = 1, \cdots, T \). Thus in \( (P_{sto}) \) the minimization could be restricted to those \( x \in \mathcal{N} \) that satisfy these constraints, rather than over all of \( \mathcal{N} \).

We have already mentioned earlier that the information process is a significant feature of the stochastic version \( (P_{sto}) \) of our model. We have modeled it here by an increasing sequence of \( \sigma \)-fields \( \zeta_t \) for \( t = 0, \cdots, T \). Each \( \zeta_t \) represents the field generated by the information-events accessible to the decision maker in time period \( t \). We implicitly assume that there is no loss of information from one time period to the next, since for all \( t \) we have \( \zeta_{t-1} \subset \zeta_t \). To gauge the flexibility of this modeling of the information process, it is convenient to introduce the increasing sequence of \( \sigma \)-fields \( F_t \subset \mathcal{A} \), for all \( t = 0, \cdots, T \). Each \( F_t \) is the \( \sigma \)-field generated by the random events that occur before or at time \( t \), where \( F_T = \{ x = (x_0, \cdots, x_T) \mid x_t \text{ is } F_t \text{ measurable} \} \).

Our aim here is to treat the analog of this model in discrete time, imposing only lower semicontinuous (lsc for short) or Lipschitzian assumptions.

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2 Definitions and preliminaries

In the next section, although our necessary optimality conditions could be given with the use of many types of subdifferentials, we will limit ourselves to state and establish them with the basic limiting subdifferential see [14].

Recall first that for a proper lsc function \( f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\} \) and \( u \in \text{dom} f \) the Fréchet subdifferential \( \partial f(u) \) is defined by the fact that a vector \( v \in \partial f(u) \) when for any positive number \( \varepsilon \) there exists some positive number \( \eta \) such that one has

\[
\langle v, u' - u \rangle \leq \varepsilon \|u' - u\| \quad \text{for all } u' \in B(u, \eta),
\]

where \( B(u, \eta) \) denotes the open ball with radius \( \eta \) centered at the point \( u \). One puts in general \( \partial f(u) = \emptyset \) when \( f(u) \) is not finite.

When \( f \) is the indicator function \( \delta_S \) of a closed subset \( S \subset \mathbb{R}^n \), that is, \( \delta_S(u) = 0 \) if \( u \in S \) and \( \delta_S(u) = +\infty \) otherwise, its Fréchet subdifferential at a point \( u \in S \) is a cone. It is generally called the Fréchet normal cone to \( \partial f(u) \). Modifying slightly the definition above, we say that a vector \( v \) belongs to the limiting subdifferential \( \partial f(u) \) at a point \( u \in \text{dom} f \) when there exists a sequence \( (u_k, f(u_k)) \) converging to \( (u, f(u)) \) and vectors \( v_k \in \partial f(u_k) \) with \( v_k \to v \). As above, one sets \( \partial f(u) = \emptyset \) if \( u \notin \text{dom} f \). The set \( \partial f(u) \) is nonconvex in general but it enjoys full point based calculus rules. For example, if \( g : \mathbb{R}^n \to \mathbb{R} \) is a locally Lipschitz function one has [13] and [21] the inclusion

\[
\partial(f + g)(u) \subset \partial f(u) + \partial g(u),
\]

where the addition in the second member is taken in the usual Minkowski sense, that is, \( \partial f(u) + \partial g(u) := \{v + v' | v \in \partial f(u), v' \in \partial g(u)\} \).

The inclusion (6) can be also obtained under a much weaker condition than the local Lipschitz property of one of the functions \( f \) and \( g \). To see that, let us recall the concept of singular limiting subdifferential. Modifying slightly the definition above, we say that a vector \( v \) belongs to the singular limiting subdifferential \( \partial^\infty f(u) \) at a point \( u \in \text{dom} f \) when there exists a sequence \( (u_k, f(u_k)) \) converging to \( (u, f(u)) \), positive numbers \( \lambda_k \downarrow 0 \) and vectors \( v_k \in \partial f(u_k) \) such that \( \lambda_k v_k \to v \). So, if for two lsc functions \( f, g \) the qualification condition \( \partial^\infty f(u) \cap -\partial^\infty g(u) = \{0\} \) holds, then one has see [13, 21] \( \partial(f + g)(u) \subset \partial f(u) + \partial g(u) \). This qualification condition can be translated see [13, 21] in the case of any finite number of lsc functions: for a finite number of lsc functions \( f_i, i = 0, 1, \ldots, m \), and for \( u \in \bigcap_{i=0}^m \text{dom} f_i \) one has

\[
\partial\left(\sum_{i=0}^m f_i\right)(u) \subset \sum_{i=0}^m \partial f_i(u),
\]

whenever for any \( y_i \in \partial^\infty f_i(u) \) with \( \sum_{i=0}^m y_i = 0 \) one necessarily has \( y_0 = y_1 = \cdots = y_m = 0 \).

The inclusion (6) is a particular case of (7) since

\[
\partial^\infty g(u) = \{0\} \quad \text{whenever } f \text{ Lipschitz near } u.
\]

The same qualification condition above also gives see [13, 21]

\[
\partial^\infty\left(\sum_{i=0}^m f_i\right)(u) \subset \sum_{i=0}^m \partial^\infty f_i(u).
\]

Concerning the composition operation, we will recall the result with the composition with a linear mapping. If \( A : \mathbb{R}^m \to \mathbb{R}^n \) is a linear surjective mapping, then see [13, 21]

\[
\partial(f \circ A)(u) \subset A^*\partial f(Au) \quad \text{and} \quad \partial^\infty(f \circ A)(u) \subset A^*\partial^\infty f(Au),
\]
where $A^*$ denotes the adjoint of $A$ and $A^*\partial f(Au) := \{A^*v \mid v \in \partial f(Au)\}$.

As for the Fréchet normal cone (see above), the limiting normal cone to a closed subset $S$ at $u \in S$ is defined through its indicator function by $N_S(u) := \partial \delta_S(u)$. Sometimes one write $N(S, u)$ in place of $N_S(u)$. The connexion with the singular subdifferential is provided by the equalities
\[ \partial^\infty \delta_S(u) = \partial \delta_S(u) = N(S, u). \]

Of course, when the point $u$ is a minimum point for the function $f$ one has both $0 \in \partial f(u)$ and $0 \in \partial f(u)$, the first inclusion being obvious under the minimum point assumption and the second one being a consequence of the fact that one always has $\partial f \subset \partial f$. Further, when $f$ is convex, the Fréchet subdifferential and the limiting subdifferential coincide with the usual Fenchel subdifferential of Convex Analysis.

In the next section, we will just say subdifferential of $f$ and normal cone to $S$ in place of limiting subdifferential of $f$ and limiting normal cone to $S$.

And also we introduce the concept of Clarke’s subdifferential. Firstly we need to define the concept of tangent cone of Clarke. In all the rest $X$ is a real Banach space. Let $C$ a nonempty closed subset of $X$ and $\bar{x}$ a point of $C$. We said that a vector $v \in X$ is in the tangent cone of Clarke to $C$ at the point $\bar{x}$ see [5, 6], and we write $v \in T_C(\bar{x}, \bar{x})$ when there exist some sequences $t_n \downarrow 0$, $x_n \xrightarrow{C} \bar{x}$ et $v_n \rightarrow v$ such that for all $n \in N$ we have $x_n + t_nv_n \in C$. The normal cone of Clarke to $C$ at the point $\bar{x} \in C$ is given by the negative polar cone of tangent cone, that means
\[ N_C(\bar{x}) = (T_C(\bar{x}, \bar{x}))^0 = \{x^* \in X^* \mid \langle x^*, v \rangle \leq 0, \forall v \in T_C(\bar{x}, \bar{x})\}. \]

Let now $f : X \rightarrow \mathbb{R} \cup \{+\infty\}$ a function l.s.c and $\bar{x}$ a point where $f$ is finite. considering the normal cone to the epigraph of $f$ we can define the Clarke subdifferential of $f$ at $\bar{x}$ by
\[ \partial_C f(\bar{x}) := \{x^* \in X^* \mid (x^*, -1) \in N_C(\text{epi} f, (\bar{x}, f(\bar{x}))\}. \]

We define also the singular subdifferential of Clarke by
\[ \partial^\infty_C f(\bar{x}) := \{x^* \in X^* \mid (x^*, 0) \in N_C(\text{epi} f, (\bar{x}, f(\bar{x}))\}. \]

As usual, we put $\partial_C f(\bar{x}) = 0 = \partial^\infty_C f(\bar{x})$ when $f(\bar{x})$ is not finite. Contrary to limiting subdifferential, the Clarke subdifferential can be found through its suprot function see [17, 5]. For each vector $v \in X$ and $\bar{x} \in \text{dom} f$, we consider the general directional derivative of Rockafellar at the direction $v$ define by
\[ d^1(\bar{x}; v) := \limsup_{x \rightarrow \bar{x}} \inf_{v' \rightarrow v} t^{-1}[f(x + tv') - f(x)], \]

with
\[ \limsup_{x \rightarrow \bar{x}} \inf_{v' \rightarrow v} \psi(x, v') = \inf_{\epsilon > 0} \inf_{\eta > 0} \inf_{\|x - \bar{x}\| < \epsilon} \sup_{\|v' - v\| < \eta} \psi(x, v'). \]

Rockafellar see [17] had proved that this directional derivative is a function of $v$ which is sublinear and l.s.c and it was also proved that the Clarke subdifferential is characterize by
\[ \partial_C f(\bar{x}) = \{x^* \in X^* \mid \langle x^*, v \rangle \leq d^1 f(\bar{x}; v) \forall v \in X\}. \]

Concerning the calculus rules, we start by considering the case of the composition with a linear continuous surjective mapping. Let $A : Z \rightarrow X$ a linear continuous surjective mapping and $z \in Z$ with $Az \in \text{dom} f$. Then
\[ \partial_C (f \circ A)(z) = A^* \partial_C f(Az). \]
Remark We will establish the optimality necessary conditions of those models in two contexts, the first one is when the Banach $X$ is a Banach space which is $L^\infty(\Omega, A, \mu, (\mathbb{R}^n)^{T+1})$, and in the second context $X$ is an Asplund space which is $L^p(\Omega, A, \mu, (\mathbb{R}^n)^{T+1})$, $1 < p < \infty$. We have already remind the concept of the limiting subdifferential and a tot of basic elements in the context of Asplund space. Outside the Asplund spaces, the limiting subdifferential, for all locally Lipschitzian functions, also can be empty at all point of the domain. For the spaces which are not Asplund spaces, the limiting subdifferential has not the calculus rules. So in this case we will use the Clarke subdifferential.

3 Necessary optimality conditions with Clarke subdifferential

The following theorem deals with the model $(P_{sto})$, that is, the case of Lipschitzian functions $l$ and $L_t$, explicit set constraint $C$ and set-valued mapping constraint $F_t$.

3.1 Theorem

**Theorem 1** Let $\bar{x} \in \mathcal{N} \subset L^\infty(\Omega, A, \mu, (\mathbb{R}^n)^{T+1})$ be a solution of model $(P_{sto})$. Assume that the field $\varsigma_t = \varsigma_{t-1}$ and $l$, $L_t$ are locally Lipschitzian for all $t = 1, \cdots, T$

Then there exists some $p = (p_0, \cdots, p_T) \in L^1(\Omega, A, \mu, (\mathbb{R}^n)^{T+1})$ such that:

a) $\beta_0, -\beta_T \in \partial l(E\bar{x}_0, E\bar{x}_T) + N_C(E\bar{x}_0, E\bar{x}_T)$; $\beta_0 = (E^0 p_0)(\omega)$, $\beta_T = (E^T p_T)(\omega)$.

b) $(E^t \Delta p_t)(\omega), (E^t p_t)(\omega)) \in \partial L_t(\omega, x_{t-1}(\omega), \Delta \bar{x}_t(\omega)) + N_{S_t(\omega)}(x_{t-1}(\omega), \Delta \bar{x}_t(\omega))$

\forall t = 1, \cdots, T - 1 a.s.

Where $S_t \equiv \text{gph} F_t$.

**Proof** Step 1. Consider the function $\Phi : L^\infty(\Omega, A, \mu, (\mathbb{R}^n)^{T+1}) \to \mathbb{R}$

$$\Phi(x) = l(E\bar{x}_0, E\bar{x}_T) + k_1 d_C(E\bar{x}_0, E\bar{x}_T) + E \sum_{t=1}^{T} (L_t(., x_{t-1}, \Delta x_t) + k_2 d_{S_t}(x_{t-1}, \Delta x_t)), k_1 > 0.$$

And put

$$\Phi_0(x) = l(E\bar{x}_0, E\bar{x}_T) + k_1 d_C(E\bar{x}_0, E\bar{x}_T) = (l \circ B_0)(x) + k_1 (d_C \circ B_0)(x)$$

$$\Phi_t(x) = L_t(., x_{t-1}, \Delta x_t) + k_2 d_{S_t}(x_{t-1}, \Delta x_t), \forall t = 1, \cdots, T.$$

Then

$$(E\Phi_t)(x) = EL_t(., x_{t-1}, \Delta x_t) + k_2 Ed_{S_t}(x_{t-1}, \Delta x_t) = I_t(x) + k_2(E(d_{S_t \circ B_t}))(x), k_2 > 0.$$

Where

$$B_0 : L^\infty(\Omega, A, \mu, (\mathbb{R}^n)^{T+1}) \to \mathbb{R}^n \times \mathbb{R}^n$$

$$x \mapsto B_0x = (E\bar{x}_0, E\bar{x}_T)$$

And $B_t = (x_{t-1}, \Delta x_t), \forall t = 1, \cdots, T$

$$I_t(x) = \int_{\Omega} L_t(\omega, x_{t-1}(\omega), \Delta x_t(\omega))d\mu(\omega) = \int_{\Omega} L_t(\omega, B_t x(\omega))d\mu(\omega)$$

hence $I_t(x) = (\varsigma_t \circ B_t)(x)$
Where $\mathcal{Z}(u,v) = \int_\Omega L_t(u,\nu, v(\nu))d\mu(\omega), \forall (u,v) \in L^\infty(\Omega,A,\mu,(\mathbb{R}^n)^2)$.

Then $\Phi(x) = \Phi_0(x) + E\sum_{t=1}^T \Phi_t(x)$

$$= (l \circ B_0)(x) + (d_C \circ B_0)(x) + \sum_{t=1}^T (E\Phi_t)(x)$$

hence

$$\Phi(x) = (l \circ B_0)(x) + k_1(d_C \circ B_0)(x) + \sum_{t=1}^T ((\mathcal{Z}_t \circ B_t)(x) + k_2E(d_S_i \circ B_t)(x)).$$

$\bar{x} \in \mathcal{N}$ is a solution of $(P_{sto})$ then

$$0 \in \partial \Phi(\bar{x}) = \partial(\Phi_0 + \sum_{t=1}^T (E\Phi_t))(\bar{x})$$

As the functions $\Phi_t$ are locally Lipschitzians for all $t = 0, \cdots, T$ because $l$, $L_t$, $d_C$ and $d_S_i$ also for all $t = 1, \cdots, T$.

Then $0 \in \partial \Phi(\bar{x}) \subset \partial \Phi_0(\bar{x}) + \sum_{t=1}^T \partial(E\Phi_t)(\bar{x}).$

As the function $l$ is locally Lipschitzian and the mapping $B_0$ is surjective, according to the calculus rule of subdifferential of composition functions in (10), we have

$$\partial \Phi_0(\bar{x}) \subset \partial(l \circ B_0)(\bar{x}) + k_1\partial(d_C \circ B_0)(\bar{x}) \subset B_0^*\partial l(B_0\bar{x}) + k_1B_0^*\partial d_C(B_0\bar{x})$$

And as $L_t$ is locally lipschitzian function for all $t = 1, \cdots, T$ then

$$\partial(E\Phi_t)(\bar{x}) \subset \partial(\mathcal{Z}_t \circ B_t)(\bar{x}) + k_2\partial(E(d_S_i \circ B_t))(\bar{x})$$

which ensures the existence of some

$$\zeta_0^1 \in B_0^*\partial l(B_0\bar{x})$$

$$\zeta_0^2 \in B_0^*\partial d_C(B_0\bar{x})$$

$$\zeta_t^1 \in \partial(\mathcal{Z}_t \circ B_t)(\bar{x}) \subset B_t^*\partial \mathcal{Z}_t(B_t\bar{x}), \forall t = 1, \cdots, T.$$  

$$\zeta_t^2 \in \partial(E(d_S_i \circ B_t))(\bar{x}), \forall t = 1, \cdots, T.$$

Such that $\sum_{t=0}^T (\zeta_t^1 + \zeta_t^2) = 0$.

And some

$$\bar{v}_1^0 = (\bar{v}_1^0, \bar{v}_2^0) \in \partial l(E\bar{x}, E\bar{x})$$

such that $\zeta_0^1 = B_0^*\bar{v}_1^0$

And $\bar{v}_2^0 = (\bar{v}_1^0, \bar{v}_2^0) \in \partial d_C(E\bar{x}, E\bar{x}) \subseteq N_C(E\bar{x}, E\bar{x})$ such that $\zeta_0^2 = B_0^*\bar{v}_2^0$.

Now we must calculate $B_0^*$. Indeed let $z \in \mathbb{R}^n \times \mathbb{R}^n$

$$(B_0^*z, h)_{\mathcal{M}} = \int_\Omega <(z_1,z_2);(Eh_0,Eh_T)>_{(\mathbb{R}^n)^2}d\mu(\omega)$$

$$= \int_\Omega <z_1,Eh_0> d\mu(\omega) + \int_\Omega <z_2,Eh_T> d\mu(\omega)$$

$$= \int_\Omega <z_1, \int_\Omega h_0(\omega)d\mu(\omega)> d\mu(\omega) + \int_\Omega <z_2, \int_\Omega h_T(\omega)d\mu(\omega)> d\mu(\omega)$$
we have 

\[
\int_\Omega \langle \int_\Omega \xi_{1,2} d\mu(\omega), h(\omega) \rangle d\mu(\omega) + \int_\Omega \langle \int_\Omega \xi_{2,3} d\mu(\omega), h(\omega) \rangle d\mu(\omega)
\]

By the calculate of Clark such that 

\[
L_\Omega \exists \zeta_t \in B_t^* \partial S_t(B_t \bar{x}), \forall t = 1, \cdots, T
\]

which gives the existence of some

\[
\zeta_t^1 = B_t^* \partial S_t(B_t \bar{x})
\]

such that

\[
\zeta_t^2 = B_t^* \partial S_t(B_t \bar{x})
\]

such that \( \zeta_t^2(\omega) = B_t^* \partial S_t(B_t \bar{x}) a.s. \)

As \( B_t^* (z_1, z_2) = (0, \cdots, 0, z_1 - z_2, z_2, 0, \cdots, 0), \forall (z_1, z_2) \in L^1(\Omega, A, \mu, (R^n)^2) \)

We have

\[
\zeta_t^1 = (0, \cdots, 0, \varphi_{t,1}^1 - \varphi_{t,2}^1, \varphi_{t,1}^2, 0, \cdots, 0) \text{ and } \zeta_t^2(\omega) = (0, \cdots, 0, \varphi_{t,2}^1(\omega), \varphi_{t,2}^2(\omega), 0, \cdots, 0) a.s. \forall t = 1, \cdots, T.
\]

And hence

\[
\zeta_t^0 = (E \varphi_{t,1}^1, 0, \cdots, 0, E \varphi_{t,2}^1)
\]
\[
\begin{align*}
\zeta_0^2 &= (E\vartheta_{0,1}^2, 0, \ldots, 0, E\vartheta_{0,2}^2) \\
\zeta_1^1 &= (0, \ldots, 0, \vartheta_{1,1}(\omega) - \vartheta_{1,2}(\omega), \vartheta_{1,2}(\omega), 0, \ldots, 0) \\
\zeta_T^2(\omega) &= (0, \ldots, 0, \vartheta_{T,1}(\omega) - \vartheta_{T,2}(\omega), \vartheta_{T,2}(\omega), 0, \ldots, 0) \text{ a.s.} \forall t = 1, \ldots, T.
\end{align*}
\]

Such that
\[
\sum_{t=0}^{T} (\zeta_t^1 + \zeta_t^2) = 0
\]
which means
\[
\begin{align*}
E\vartheta_{0,1}^1 + E\vartheta_{0,1}^2 + \vartheta_{1,1}(\omega) - \vartheta_{1,2}(\omega) + \vartheta_{1,2}(\omega) - \vartheta_{1,2}(\omega) &= 0 \ldots (\bar{1}) \\
\vartheta_{1,2}(\omega) + \vartheta_{2,1}(\omega) - \vartheta_{1,2}(\omega) + \vartheta_{2,1}(\omega) - \vartheta_{2,2}(\omega) &= 0 \\
\vdots
\end{align*}
\]
\[
\begin{align*}
\vartheta_{T-1,2}(\omega) + \vartheta_{T-2,1}(\omega) - \vartheta_{1,2}(\omega) + \vartheta_{2,1}(\omega) - \vartheta_{2,2}(\omega) &= 0 \forall t = 2, \ldots, T - 1 \ldots (\bar{2})
\end{align*}
\]
\[
E\vartheta_{0,2}^1 + E\vartheta_{0,2}^2 + \vartheta_{T,2}(\omega) + \vartheta_{T,2}(\omega) &= 0 \ldots (\bar{3}).
\]

Putting \(p = (p_0, \ldots, p_T) \in L^1(\Omega, \lambda, (\mathbb{R}^n)^{T+1})\) such that
\[
(E^0p_0)(\omega) = E\vartheta_{0,1}^1 + E\vartheta_{0,1}^2 = E(\vartheta_{0,1}^1 + \vartheta_{0,1}^2) = \beta_0.
\]
And
\[
(E^t\varphi_t)(\omega) = \vartheta_{T,2}(\omega) + \vartheta_{T,2}(\omega) \forall t = 1, \ldots, T.
\]
then \(E^0p_0 = \vartheta_{T,2}^1 + \vartheta_{T,2}^2 \text{ a.s.}\) And
\[
(E^T p_T)(\omega) = \vartheta_{T,2}^1 + \vartheta_{T,2}^2(\omega) = -E\vartheta_{0,2}^1 - E\vartheta_{0,2}^2 = -E(\vartheta_{0,2} - \vartheta_{0,2}^2).
\]
Hence by \((\bar{3})\) \((E^T p_T)(\omega) = -E(\vartheta_{0,2} + \vartheta_{0,2}^2) = \beta_T \text{ a.s.}\) And
\[
(E^t \Delta p_t)(\omega) = (E^t p_t)(\omega) - (E^t p_{t-1})(\omega) = \vartheta_{T,2}^1 + \vartheta_{T,2}^2(\omega) - (E^t p_{t-1})(\omega)
\]
and by \((\bar{2})\) we have
\[
(E^t \Delta p_t)(\omega) = \vartheta_{T-1,2}(\omega) + \vartheta_{T-2,2}(\omega) + \vartheta_{T-1,1}(\omega) + \vartheta_{T-2,1}(\omega) - (E^t p_{t-1})(\omega).
\]
So \((E^t \Delta p_t)(\omega) = (E^{t-1} p_{t-1})(\omega) - (E^t p_{t-1})(\omega) + \vartheta_{T-1,1}(\omega) + \vartheta_{T-2,1}(\omega) - (E^t p_{t-1})(\omega) = -(E^{t-1} p_{t-1})(\omega) + \vartheta_{T-1,1}(\omega) + \vartheta_{T-1,1}(\omega) \text{ a.s.}\)

With hypothesis we have the field \(\varsigma_t = \varsigma_t, \forall t = 2 \cdots T\) which means that the \(E^t\Delta t\) terms can always be dropped and hence \((E^t t p_{t-1})(\omega) = 0 \text{ a.s.}\),
So we have \((E^t \Delta p_t)(\omega) = \vartheta_{T-1,1}(\omega) + \vartheta_{T-1,1}(\omega) \text{ a.s.}\)

Then we conclude that
Theorem 2 Let $\bar{x} \in \mathcal{N} \subset L^p(\Omega, A, \mu, (\mathbb{R}^n)^{T+1})$, $1 < p < \infty$ be a solution of model $(P_{sto})$ and Assume that the field $\varsigma = \varsigma_- (\cdot)$ and $l_t$ are proper and lsc for all $t = 1, \ldots, T$ and that the following qualification condition $\tilde{Q}(\bar{x}(\omega))$ holds:

\[
\begin{align*}
&\left\{ \begin{array}{l}
 y_t \in l^q(\Omega, A, \mu, (\mathbb{R}^n)^{T+1}) : (\alpha_0, -\alpha_T) \in \partial^\infty I_C(E\bar{x}_0, E\bar{x}_T), \alpha_0 = p_0(\omega), \alpha_T = p_T(\omega) \\
 ((E^t \Delta y_t)(\omega), (E^t y_t)(\omega)) \in \partial^\infty I_{S_t}(x_{t-1}(\omega), \Delta \bar{x}_t(\omega)) a.s
\end{array} \right.
\end{align*}
\]

Then there exists some $p = (p_0, \ldots, p_T) \in L^p(\Omega, A, \mu, (\mathbb{R}^n)^{T+1})$ satisfying relations (a) and (b) of previous theorem.

**Proof** We follow the same steps as in the proof of precedent Theorem. And then we redo the same proof with the space $L^p(\Omega, A, \mu, (\mathbb{R}^n)^{T+1})$ and with the limiting subdifferential. Finally found similar results. □

Remark The prospect of studying stochastic models of Bolza type in continuous time, as limits of sequences of discrete time problems, provided some of the motivation for this study.

**References**


