Quasilinear and singular systems: the cooperative case

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Abstract

We investigate the following quasilinear and singular system,

\[
\begin{align*}
-\Delta_{p_1} u &= u^{\alpha_1} v^{\beta_1} \quad \text{in } \Omega \\
-\Delta_{p_2} v &= u^{\alpha_2} v^{\beta_2} \quad \text{in } \Omega \\
u > 0, v > 0, \quad u, v &= 0 \text{ on } \partial \Omega,
\end{align*}
\]

where \(\Omega\) is an open bounded domain with smooth boundary, \(1 < p_i < \infty\) and \(\alpha_i + \beta_i < 0\) for any \(i = 1, 2\). We employ monotone methods in order to show the existence of a unique (positive) solution of problem \((P)\) in some cone. When \(\alpha_i + \beta_i > -1\) for \(i = 1, 2\), we prove a regularity result for solutions to problem \((P)\) in \(C^{1,\beta}(\Omega)\) with some \(\beta \in (0, 1)\). Furthermore, we show that \(\min_{i=1,2} \alpha_i + \beta_i > -1\) is a reasonable sufficient (and likely optimal) condition to obtain solutions of problem \((P)\) in \(C^{1}(\Omega)\). Keywords: quasilinear cooperative systems, singular equations, sub and supersolutions.

1 Introduction

This paper deals with the existence and uniqueness of a solution of the problem:

\[
\begin{align*}
-\Delta_{p_1} u &= u^{\alpha_1} v^{\beta_1} \quad \text{in } \Omega \\
-\Delta_{p_2} v &= u^{\alpha_2} v^{\beta_2} \quad \text{in } \Omega \\
u > 0, v > 0, \quad u, v &= 0 \text{ on } \partial \Omega,
\end{align*}
\]

where \(\Omega\) is a bounded domain in \(\mathbb{R}^N\) with smooth boundary, \(\alpha_i + \beta_i < 0\) for \(i = 1, 2\) and \(\Delta_p w = \text{div} \left( |\nabla w|^{p-2} \nabla w \right)\) is the \(p\)-Laplacian operator with \(1 < p < \infty\). Throughout this paper we make the following hypothesis on the parameters:

\[
\alpha_2, \beta_1 > 0,
\]

which means that system \((1)\) is cooperative. The aim of this paper is to find the existence and uniqueness of weak solutions to \((1)\).

Following the pioneering work Crandall-Rabinowitz-Tartar [6], existence, uniqueness and/or multiplicity and regularity of positive solutions to some semilinear singular elliptic equations have been studied in the last thirty years by using different methods of nonlinear partial differential equations and nonlinear functional analysis. In particular, extensions of the method of sub and supersolutions and variational methods have been used for this study. Most of these results are collected in the survey Hernandez-Mancebo [17], Hernandez-Mancebo-Vega [18] and the book Ghergu-Radulescu [14] (and the references therein). General regularity results are obtained in Gui-Hua Lin [16]. The case of singular elliptic systems has been much less considered in the literature. However, rather general results for both cooperative and non-cooperative elliptic systems, together with some examples, are given in Hernandez-Mancebo-Vega [19] (see also the references there for previous works). Existence is proved by using sub and supersolutions and applying the results proved for one single equation in [17] and [18]. Uniqueness is proved as well under some "concavity" assumption. These results do not include the strongly singular case (as it is defined below). Recently existence and uniqueness for a class of semilinear non-cooperative systems were obtained in Ghergu [13] where the strongly singular case was also studied. Existence is proved there by using Schauder's fixed point Theorem. The case where the classical Laplace operator is replaced by the p-Laplacian type operators has not been considered so far. However concerning the case of one single equation, we can quote the results in Aranda-Godoy [3], Giacomoni-Schindler-Takac [15] and Perera-Silva [22]. In El Manouni-Perera-Shivaji [21] the authors study by an approximation procedure a class of quasilinear cooperative systems under less general assumptions than ours in the present paper. Furthermore, the results in [21] do not deal with strongly singular nonlinearities and no uniqueness result is obtained. Concerning the uniqueness of solutions, the point is that equations involving quasilinear elliptic operators yield additional difficulties for obtaining the validity of the strong comparison principle (see Cuesta-Takac [7], Fleckinger-Takac [11], Giacomoni-Schindler-Takac [15] and Vazquez [25]) which requires the $C^{1}$-regularity for solutions. In the non singular case, the local $C^{1,\alpha}$-regularity and regularity up to the boundary of solutions to p-Laplacian equations were obtained in DiBenedetto [4], Lieberman [20] and Tolksdorf [24]. In this regard, the $C^{1,\alpha}$-regularity can be used to get multiplicity results and the extension of Brezis-Nirenberg [5] result to the degenerate case (see Garcia-Azorero-Manfredi-Peral [12] and [15]). In case where the $C^{1}$-regularity cannot be used (in the strongly singular case for instance) for proving uniqueness results, we appeal a suitable modification of Krasnoleskii’s argument developed in Section 3.

2 Main results

We will consider separately the case when the exponents $\alpha_{1}$, $\alpha_{2}$, $\beta_{1}$ and $\beta_{2}$ are such that

$$-1 < \alpha_{i} + \beta_{i} < 0, \quad i = 1, 2.$$  \hspace{1cm} (3)

and the case when system (1) is "strongly singular"

$$\alpha_{i} + \beta_{i} < -1, \quad i = 1, 2 \text{ with } \beta_{1} < p_{1} - 1 \text{ and } \alpha_{2} < p_{2} - 1.$$  \hspace{1cm} (4)

Our first main result is given by the theorem below dealing with the first case:

**Theorem 1** Let (2) and (3) be satisfied. Then system (1) possesses a solution $(u, v)$ in $W^{1, p_{1}}_{0}(\Omega) \times W^{1, p_{2}}_{0}(\Omega)$ such that

$$u, v \geq c \text{ dist } (x, \partial \Omega),$$

for some positive constant $c$.  \hspace{1cm} (5)
By the Hardy Inequality, the cone condition given in (5) provides that the right hand sides in (1) belong to the dual spaces $W^{-1,p_i'}(\Omega)$ and $W^{-1,p_2}(\Omega)$ respectively. Then, we can prove the following result:

**Theorem 2** Let (2) and (3) be satisfied. Then there exists a unique weak solution $(u, v)$ in

$$C_1 \overset{def}{=} \{(w_1, w_2) \in W^{1,p_1}_0(\Omega) \times W^{1,p_2}_0(\Omega) / (w_1, w_2) \text{ satisfies } (5)\}.$$

Moreover, $(u, v) \in C^{1,\delta_1}(\Omega) \times C^{1,\delta_1}(\Omega)$ where for any $i = 1, 2$

$$\delta_i = \begin{cases} 1 + (\alpha_i + \beta_i) & \text{if } p_i \geq 2, \\ 1 + \frac{p_i}{2}(\alpha_i + \beta_i) & \text{if } p_i \geq 2. \end{cases}$$

We denoted by $\phi_1$ (resp. $\psi_1$) the normalized positive eigenfunction associated with the principal eigenvalue $\lambda_1(\Omega, p_1)$ (resp. $\lambda_1(\Omega, p_2)$) of $-\Delta_{p_1}$ (resp. $-\Delta_{p_2}$):

$$-\Delta_{p_i}\phi_1 = \lambda_1(\Omega, p_1)|\phi_1|^{p_i-2}\phi_1 \text{ in } \Omega; \quad \phi_1 = 0 \text{ on } \partial \Omega$$

and

$$-\Delta_{p_2}\psi_1 = \lambda_1(\Omega, p_2)|\psi_1|^{p_2-2}\psi_1 \text{ in } \Omega; \quad \psi_1 = 0 \text{ on } \partial \Omega.$$

From Annane [1] we have that $\phi_1, \psi_1 \in C^{1,\delta}(\Omega)$ for any $\delta < 1$ and satisfy (5). Then, the existence result in the strongly singular case can be formulated as follows.

**Theorem 3** Assume that conditions (2) and (4) are satisfied. Then problem (1) possesses at least one weak solution $(u, v)$ satisfying for some constants $c_1, c_2 > 0$

$$c_1(\phi_1^\gamma, \psi_1^\theta) \leq (u, v) \leq c_2(\phi_1^\gamma, \psi_1^\theta),$$

where $\gamma, \theta \in (0, 1)$ are the unique solutions to the following linear system

$$\begin{cases} (\gamma - 1)(p_1 - 1) - 1 = \alpha_1\gamma + \beta_1\theta \\ (\theta - 1)(p_2 - 1) - 1 = \alpha_2\gamma + \beta_2\theta. \end{cases}$$

Using the cone condition given in (6), we can prove the following uniqueness result:

**Theorem 4** Assume that conditions (2) and (4) are satisfied. The weak solution $(u, v)$ given in Theorem 3 belongs to $W^{1,p_1}_0(\Omega) \times W^{1,p_2}_0(\Omega)$ if and only if the following condition is satisfied

$$-(\alpha_i\gamma + \beta_i\theta) < 2 - \frac{1}{p_i}$$

for $i = 1, 2$. If (8) holds, problem (1) admits a unique weak solution $(u, v)$ satisfying (6).

Similar results can be proved in the more general case where $\alpha_i + \beta_i \leq -1$. In particular, we only state the following result which can be proved by similar arguments.

**Theorem 5** Let us assume conditions (2) and $\alpha_i + \beta_i = 1$ for $i = 1, 2$. Then there exists a unique solution $(u, v)$ to (1) in $W^{1,p_1}_0(\Omega) \times W^{1,p_2}_0(\Omega)$ satisfying $(u, v) \geq -c_\varepsilon \left(\phi_1(\ln \phi_1)^{1+\varepsilon}, \psi_1(\ln \psi_1)^{1+\varepsilon}\right)$ for any $\varepsilon > 0$ and for some $c_\varepsilon > 0$ depending on $\varepsilon > 0$.

The outline of the paper is as follows. We deal first with the case where $\alpha_i, \beta_i$ for $i = 1, 2$ satisfy (3). We prove Theorem 1 by an appropriated weak comparison principle. The uniqueness and regularity results given by Theorem 2 use some arguments from Giacomoni-Schindler-Takac [15]. Then we deal with the case where $\alpha_i, \beta_i$ for $i = 1, 2$, satisfy (4). We prove Theorem 3 by using an approximated problem for which we introduce appropriated sub and supersolutions. These barrier functions provide the control of the solutions near the boundary. From this control, we establish Theorem 4.
3 Preliminaries

In this section we present some notations and results needed in the sequel. Given $1 \leq p_i < \infty$ ($i = 1, 2$), the norm in $L^{p_i}(\Omega)$ is denoted by

$$\|u\|_{p_i} = \left(\int_{\Omega} |u|^{p_i}\right)^{1/p_i}$$

and the norm in $W^{1,p_i}_0(\Omega)$ by

$$\|u\|_{W^{1,p_i}_0(\Omega)} = \left(\int_{\Omega} |\nabla u|^{p_i}\right)^{1/p_i}.$$ 

We recall that is the dual space of . As usual, $C^\infty_0(\Omega)$ denotes the space of all $C^\infty$ functions $f: \Omega \to \mathbb{R}$ with compact support and the function $d(x)$ the distance from a point $x \in \overline{\Omega}$ to the boundary $\partial \Omega$, where $\overline{\Omega} = \Omega \cup \partial \Omega$ is the closure of $\Omega \subset \mathbb{R}^N$. This means that

$$d(x) = \text{dist}(x, \partial \Omega) = \inf_{y \in \partial \Omega} |x - y|.$$ 

Here, for $s \in \mathbb{R}$, we denote by $s^\pm = \max(\pm s, 0)$ so that $s = s^+ - s^-$. A pair of nonnegative functions $(u, v)$ in $W^{1,p_1}_{\text{loc}}(\Omega) \times W^{1,p_2}_{\text{loc}}(\Omega)$ is called a weak solution (resp. subsolution, supersolution) of (1) if

$$\begin{cases}
\int_{\Omega} |\nabla u|^{p_1-2} \nabla u \nabla \varphi - \int_{\Omega} u^{\alpha_1} v^{\beta_1} \varphi = 0 \quad (\leq 0, \geq 0) \\
\int_{\Omega} |\nabla v|^{p_2-2} \nabla v \nabla \psi - \int_{\Omega} u^{\alpha_2} v^{\beta_2} \psi = 0 \quad (\leq 0, \geq 0),
\end{cases}$$

for all $(\varphi, \psi) \in C^\infty_0(\Omega)$. 

4 Existence of weak solutions

We first consider the case where both (2) and (3) are satisfied. In this case, we can show the existence of subsolutions and supersolutions which behave like the distance function $d$ near the boundary. Precisely, we have

Lemma 6 Assume (2) and (3). Then, for $c > 0$ (resp. $C > 0$) small enough (resp. large enough) $(c\varphi_1, c\psi_1)$ (resp. $(C\varphi_1, C\psi_1)$) is a subsolution to (1) (resp. a supersolution to (1)) in $W^{1,p_1}_0(\Omega) \times W^{1,p_2}_0(\Omega)$.

Proof. Lemma 6 follows easily from the facts that $\varphi_1, \psi_1$ are positive eigenfunctions associated respectively to $\lambda_1(\Omega, p_1)$ and $\lambda_1(\Omega, p_2)$ and that $\alpha_i + \beta_i < p_i - 1$. $\square$

Under assumptions (2) and (3), we now construct by a monotone scheme a weak solution to (1) satisfying (5). Precisely, we prove the following result:

Proposition 7 Let (2) and (3) be satisfied. Then, there exists a weak solution $(u, v)$ to (1) in $C^{1,\delta}(\Omega)$ for some $0 < \delta < 1$ and such that $c(\varphi_1, \psi_1) \leq (u, v) \leq C(\varphi_1, \psi_1)$ ($c, C$ given by Lemma 6).
Let \((u, v) = (\varphi_1, \psi_1)\) and \((\pi, \nu) = (\varphi_1, \psi_1)\) \((c, C, \pi, \nu)\) as in Lemma 6. Let \((u_0, v_0) = (\pi, \nu)\). Then, for \(n \geq 1\), we define \((u_n, v_n) \in W_{0}^{1,p_1}(\Omega) \times W_{0}^{1,p_2}(\Omega)\) by the following iterative scheme:

\[
\begin{align*}
-\Delta_{p_1} u_n + K d^{p_1 + \beta_1 - (p_1 - 1)} u_{n-1}^\beta_1 &= v_{n-1}^\alpha_1 + K d^{p_1 + \beta_1 - (p_1 - 1)} u_{n-1}^\beta_1 \quad \text{in} \Omega, \\
-\Delta_{p_2} v_n + K d^{p_2 + \beta_2 - (p_2 - 1)} v_{n-1}^\beta_2 &= u_{n-1}^\alpha_2 + K d^{p_2 + \beta_2 - (p_2 - 1)} v_{n-1}^\beta_2 \quad \text{in} \Omega,
\end{align*}
\]

where \(K > 0\) is taken large enough such that the following two conditions are satisfied:

\[
\begin{align*}
K (p_1 - 1) d^{\alpha_1 + \beta_1 - (p_1 - 1) \rho_1 - 1} + \alpha_1 d^{\alpha_1 - 1} (C \psi(x))^\beta_1 &\geq 0, \quad (10) \\
K (p_2 - 1) d^{\alpha_2 + \beta_2 - (p_2 - 1) \rho_2 - 1} + \beta_2 d^{\beta_2 - 1} (C \varphi(x))^\alpha_2 &\geq 0, \quad (11)
\end{align*}
\]

uniformly in \(x \in \Omega\) and for \(t \geq \min \{u, v\}\). Note that the existence of such \(K\) (depending on \(c\)) follows from the fact that \(u\) and \(v\) satisfy (5). Hence (10) and (11) provide the validity of the weak comparison principle for the above iterative scheme in the cone

\[
\left\{ (u, v) \in W_{0}^{1,p_1}(\Omega) \times W_{0}^{1,p_2}(\Omega) / (u, v) \geq (u, v) \right\}.
\]

Then, \((u_n) \in N \text{ and } (v_n) \in N\) are monotone nonincreasing. Therefore, \((u_n) \text{ and } (v_n)\) converge pointwise to \(u\) and \(v\) respectively. Let us prove that \((u, v)\) is a weak solution to (1). From

\[
u \leq u \leq \overline{u} \text{ and } v \leq v_n \leq \overline{v},
\]

we get that

\[
l_1 d^{\alpha_1 + \beta_1} \leq u_{n}^{\alpha_1} v_{n}^{\beta_1} \leq L_1 d^{\alpha_1 + \beta_1},
\]

\[
l_2 d^{\alpha_1 + \beta_1} \leq u_{n}^{\alpha_1} v_{n}^{\beta_1} \leq L_2 d^{\alpha_1 + \beta_1},
\]

where \(l_1, l_2, L_1\) and \(L_2\) are positive constants independent of \(n\). Since \(\alpha_i + \beta_i > -1\) for \(i = 1, 2\) and from Theorem B.1 in Giacomoni-Schindler-Takac [15], we get that \(\|u_n\|_{C^{1,\delta}(\overline{\Omega})}, \|u_n\|_{C^{1,\delta}(\overline{\Omega})} \leq C\) for some \(0 < \delta < 1\) and \(C > 0\) independent of \(n\). Using Ascoli-Arzelà Theorem, \((u_n)\) and \((v_n)\) are relatively compact in \(C^{1,\delta}(\overline{\Omega})\). It follows that \((u, v)\) is a weak solution to (1). Again, using Theorem B.1 in [15], we have that \(u, v\) belong to \(C^{1,\delta}(\overline{\Omega})\) with \(\delta = \min(1 + \alpha_i + \beta_i)\) if \(p_i \geq 2\) for \(i = 1, 2\) and \(\delta = p \min(1 + \alpha_i + \beta_i)\) with \(p = \frac{\min(p_1, p_2)}{2}\).

We now consider the case where (2) and (4) are satisfied. In this case, we have the following result:

**Proposition 8** Assume that (2) and (4). Then, there exist a weak solution \((u, v)\) to (1) satisfying (6), with \(\gamma, \theta \in (0, 1)\), unique solution to the system (7).

Since \(\gamma, \theta \in (0, 1)\), the weak solution \((u, v)\) given in Proposition 8 is not in \(C^1(\overline{\Omega})^2\).

**Proof** The existence and uniqueness of \((\gamma, \theta) \in (0, 1)^2\) solution to (7) follow from (4). We show that for \(c > 0\) small enough (resp. \(C > 0\) large enough) \(c(\varphi_1^{\gamma}, \psi_1^{\theta})\) (resp. \(C(\varphi_1^{\gamma}, \psi_1^{\theta})\) is a subsolution to (1) (resp. a supersolution to (1)). Indeed, we have that

\[
-\Delta_{p_1} \varphi_1^{\gamma} = \gamma^{p_1 - 1} \left[ \lambda_1 (\Omega, p_1) (\varphi_1)^{(p_1 - 1) - (\gamma - 1)} (p_1 - 1) |\nabla \varphi_1|^{p_1 - 1} (\phi_1)^{(\gamma - 1)(p_1 - 1) - 1} \right] (13)
\]

and

\[
-\Delta_{p_2} \psi_1^{\theta} = \theta^{p_2 - 1} \left[ \lambda_1 (\Omega, p_2) (\psi_1)^{(p_2 - 1) - (\theta - 1)} (p_2 - 1) |\nabla \psi_1|^{p_2 - 1} (\psi_1)^{(\theta - 1)(p_2 - 1) - 1} \right]. (14)
\]
From the maximum principle of Vazquez [25], $|\nabla \varphi_1|, |\nabla \psi_1| \geq \eta$ near the boundary for some $\eta > 0$. Then, from (13), (14), it is easy to see that for $c > 0$ small enough (resp. $C > 0$ large enough) $c(\varphi_1^\gamma, \psi_1^{\theta})$ (resp. $C(\varphi_1^\gamma, \psi_1^{\theta})$) is a subsolution (resp. a supersolution) to (1). Moreover, if (8) holds then by the Hardy Inequality and using (13), (14) we have that $(\varphi_1^\gamma, \psi_1^{\theta}) \in W^{1,p_1}_0(\Omega) \times W^{1,p_2}_0(\Omega)$. Assuming that (8) is satisfied, let

$$
\begin{cases}
-\Delta_{p_1} u_n + K d^{\gamma(a_1 + \beta_1(-(p_1-1))) + \theta \beta_1} u^{p_1-1}_n = u^{\beta_1}_{n-1} + K d^{\gamma(a_1 + \beta_1(-(p_1-1))) + \theta \beta_1} u^{p_1-1}_{n-1} \text{ in } \Omega, \\
-\Delta_{p_1} v_n + K d^{\alpha_2 + \theta((p_2-1)} v^{p_2-1}_n = u^{\alpha_2}_{n-1} + K d^{\alpha_2 + \theta((p_2-1)} u^{p_2-1}_{n-1} \text{ in } \Omega,
\end{cases}
$$

with $(u_0, v_0) = C(\varphi_1^\gamma, \psi_1^{\theta})$ and $K$ large enough such that

$$
K(p_1 - 1) d^{\gamma(a_1 + \beta_1(-(p_1-1))) + \theta \beta_1} t^{p_1-1} + \alpha_1 t^{\alpha_1-1} (C \psi(x))^\beta_1 \geq 0, \quad (15)
$$

$$
K(p_2 - 1) d^{\alpha_2 + \theta((p_2-1)} t^{p_2-1} + \beta_2 t^{\beta_2-1} (C \varphi(x))^{\gamma \alpha_2} \geq 0. \quad (16)
$$

Again the existence of $u_n, v_n$ for $n \geq 1$ follows from the fact that at each iteration and from (17) and (18) the right hand side of the first and second equations satisfied by $u_n$ and $v_n$ belongs to $W^{-1,p_1^*}(\Omega)$ and $W^{-1,p_2^*}(\Omega)$ respectively. From (15) and (16), we get that the above iterative scheme is monotone nonincreasing provided that

$$c \varphi_1^\gamma \leq u_n \leq C \varphi_1^\gamma \quad (17)$$

and

$$c \psi_1^{\theta} \leq v_n \leq C \psi_1^{\theta}. \quad (18)$$

Again, (17) and (18) follow from the weak comparison principle and the fact that $c(\varphi_1^\gamma, \psi_1^{\theta})$ (resp. $C(\varphi_1^\gamma, \psi_1^{\theta})$) is a subsolution (resp. supersolution) to (1). Therefore, $\{u_n\}_n \in N$ (resp. $\{v_n\}_n \in N$) converge pointwise to $u$ (resp. $v$). From (17) and (18) and the Hardy inequality, we get that $\{u_n\}$, $\{v_n\}$ are uniformly bounded in $W^{1,p_1}_0(\Omega)$ and $W^{1,p_2}_0(\Omega)$ respectively. Then it is easy to derive that $(u, v)$ is a weak solution to (1). Moreover, from Lemma A.5 in Giacomoni-Schindler-Takac, we have that

$$-\Delta_{p_1}(u - 1) \leq v^{\beta_1}$$

and

$$-\Delta_{p_2}(v - 1) \leq u^{\alpha_2}.
$$

Since $\beta_1 < p_1 - 1$, $\alpha_2 < p_2 - 1$ and using classical Moser iterations (see the appendix), we get that $u, v \in L^\infty(\Omega)$ and from classical regularity theory (see Tolksdorf [23], [24]) $u, v \in C^{1,\delta}(\Omega)$ for some $0 < \delta < 1$. We now analyze the case where (8) does not hold. We cannot guarantee that the weak solution is in $W^{1,p_1}_0(\Omega) \times W^{1,p_2}_0(\Omega)$. Therefore, for $\varepsilon_1, \varepsilon_2 > 0$ we introduce the following approximated problem:

$$
\begin{cases}
-\Delta_{p_1} u = (u + \varepsilon_1)^{\alpha_1} (v + \varepsilon_2)^{\beta_1} \text{ in } \Omega, \\
-\Delta_{p_2} v = (u + \varepsilon_1)^{\alpha_2} (v + \varepsilon_2)^{\beta_2} \text{ in } \Omega, \\
u = v = 0 \text{ on } \partial \Omega
\end{cases}
$$

Let $(\gamma, \theta) \in (0, 1)^2$ be the unique solution to (7) and set for $c > 0$

$$
(u_{\varepsilon_1}, v_{\varepsilon_2}) = \left(c \left( \left( \delta_1 + \varepsilon_1^{1/\gamma} \right)^\gamma - \varepsilon_1 \right), c \left( \left( \delta_2 + \varepsilon_2^{1/\theta} \right)^\theta - \varepsilon_2 \right) \right),
$$
where $c$ is a positive constant. Since $0 < \beta_1 < p_1 - 1$, $\alpha_1 < 0$, for $c$ small enough and taking $\varepsilon_2^{1/\gamma} = \varepsilon_2^{1/\theta}$ we have

\[-\Delta_{p_1} \underline{u}_{\varepsilon_1} = c^{p_1 - 1} \phi_{1}^{p_1 - 1} \left( \lambda_1 (\Omega, p_1) \phi_{1}^{p_1 - 1} \left( \phi_{1} + \varepsilon_1^{1/\gamma} \right)^{(\gamma - 1)(p_1 - 1)} - (\gamma - 1)(p_1 - 1) |\nabla \phi_{1}|^{p_1} \left( \phi_{1} + \varepsilon_1^{1/\gamma} \right)^{(\gamma - 1)(p_1 - 1) - 1} \right) \leq c^{\beta_1} \left( \phi_{1} + \varepsilon_1^{1/\gamma} \right)^{\gamma \alpha_1} \left( \psi_{1} + \varepsilon_2^{1/\theta} \right)^{\theta \beta_1} \leq \left( c \left( \phi_{1} + \varepsilon_1^{1/\gamma} \right)^{\gamma} + (1 - c) \varepsilon_1 \right)^{\alpha_1} \left( c \left( \psi_{1} + \varepsilon_2^{1/\theta} \right)^{\theta} + (1 - c) \varepsilon_2 \right)^{\beta_1} \]

and since $0 < \alpha_2 < p_2 - 1$, $\beta_2 < 0$,

\[-\Delta_{p_2} \underline{v}_{\varepsilon_2} = c^{p_2 - 1} \phi_{2}^{p_2 - 1} \left( \lambda_2 (\Omega, p_2) \psi_{2}^{p_2 - 1} \left( \psi_{2} + \varepsilon_2^{1/\gamma} \right)^{(\theta - 1)(p_2 - 1)} - (\theta - 1)(p_2 - 1) |\nabla \psi_{2}|^{p_2} \left( \psi_{2} + \varepsilon_2^{1/\theta} \right)^{(\theta - 1)(p_2 - 1) - 1} \right) \leq c^{\alpha_2 + \beta_2} \left( \phi_{1} + \varepsilon_1^{1/\gamma} \right)^{\gamma \alpha_2} \left( \psi_{1} + \varepsilon_2^{1/\theta} \right)^{\theta \beta_2} \leq \left( c \left( \phi_{1} + \varepsilon_1^{1/\gamma} \right)^{\gamma} + (1 - c) \varepsilon_1 \right)^{\alpha_2} \left( c \left( \psi_{1} + \varepsilon_2^{1/\theta} \right)^{\theta} + (1 - c) \varepsilon_2 \right)^{\beta_2} \]

From the above inequalities, it follows that $(\underline{u}_{\varepsilon_1}, \underline{v}_{\varepsilon_2})$ is a subsolution of (19). In the same way and using $\alpha_1, \beta_2 < 0$, we can prove that for large $C > 0$

\[(\overline{u}_{\varepsilon_1}, \overline{v}_{\varepsilon_2}) = \left( C \left( \left( \phi_{1} + \varepsilon_1^{1/\gamma} \right)^{\gamma} - \varepsilon_1 \right), C \left( \left( \psi_{1} + \varepsilon_2^{1/\theta} \right)^{\theta} - \varepsilon_2 \right) \right),\]

is a supersolution to (19).

As above, let us define the sequence of functions as follows: $(u_0, v_0) = (\overline{u}_{\varepsilon_1}, \overline{v}_{\varepsilon_2})$ and, for every $n \geq 1$, $(u_n, v_n)$ is the unique solution of the following iterative system:

\[-\Delta_{p_1} u_n + K (\overline{u}_{\varepsilon_1} + \varepsilon_1)^{\alpha_1 - p_1 + 1} (\overline{v}_{\varepsilon_2} + \varepsilon_2)^{\beta_1} (u_n + \varepsilon_1)^{p_1 - 1} = (u_{n - 1} + \varepsilon_1)^{\alpha_1} (v_{n - 1} + \varepsilon_2)^{\beta_1} + K (\overline{u}_{\varepsilon_1} + \varepsilon_1)^{\alpha_1 - p_1 + 1} (\overline{v}_{\varepsilon_2} + \varepsilon_2)^{\beta_1} (u_{n - 1} + \varepsilon_1)^{p_1 - 1},\]

\[-\Delta_{p_2} v_n + K (\overline{u}_{\varepsilon_1} + \varepsilon_1)^{\alpha_2} (\overline{v}_{\varepsilon_2} + \varepsilon_2)^{\beta_2 - p_2 + 1} (v_n + \varepsilon_2)^{p_2 - 1} = (u_{n - 1} + \varepsilon_1)^{\alpha_2} (v_{n - 1} + \varepsilon_2)^{\beta_2} + K (\overline{u}_{\varepsilon_1} + \varepsilon_1)^{\alpha_2} (\overline{v}_{\varepsilon_2} + \varepsilon_2)^{\beta_2 - p_2 + 1} (v_{n - 1} + \varepsilon_2)^{p_2 - 1}\]

where $K > 0$ is taken large enough such that the following two inequalities are satisfied

\[
\alpha_1 (t + \varepsilon_1)^{\alpha_1 - 1} (\overline{v}_{\varepsilon_2} + \varepsilon_2)^{\beta_1} + K (p_1 - 1) (\overline{u}_{\varepsilon_1} + \varepsilon_1)^{\alpha_1 - p_1 + 1} (\overline{v}_{\varepsilon_2} + \varepsilon_2)^{\beta_1} (t + \varepsilon_1)^{p_1 - 2} \geq 0, \quad (20)
\]

uniformly in $x \in \Omega$ and for $t \geq \overline{u}_{\varepsilon_1}$ and

\[
\beta_2 (t + \varepsilon_2)^{\beta_2 - 1} (\overline{u}_{\varepsilon_1} + \varepsilon_1)^{\alpha_2} + K (p_2 - 1) (\overline{u}_{\varepsilon_1} + \varepsilon_1)^{\alpha_2} (\overline{v}_{\varepsilon_2} + \varepsilon_2)^{\beta_2 - p_2 + 1} (t + \varepsilon_2)^{p_2 - 2} \geq 0, \quad (21)
\]
uniformly in $x \in \Omega$ and for $t \geq \tau_{\varepsilon_2}$. From (20), (21) and the weak comparison principle, we obtain

$$\begin{align*}
(\underline{u}_{\varepsilon_1}, \underline{v}_{\varepsilon_2}) & \leq \cdots \leq (u_{n+1}, v_{n+1}) \leq (u_n, v_n) \leq \cdots \leq (u_0, v_0) = (\overline{u}_{\varepsilon_1}, \overline{v}_{\varepsilon_2}).
\end{align*}$$

From (22), we get that $(u_n, v_n)$ converge pointwise to $(u_{\varepsilon_1}, v_{\varepsilon_2}) \in C^{1,\delta}(\overline{\Omega})^2$ for some $0 < \delta < 1$ and is a solution to (19) satisfying

$$\begin{align*}
\underline{u}_{\varepsilon_1} & \leq u_{\varepsilon_1} \leq \overline{u}_{\varepsilon_1} 
\end{align*}$$

and

$$\begin{align*}
\underline{v}_{\varepsilon_2} & \leq v_{\varepsilon_2} \leq \overline{v}_{\varepsilon_2}.
\end{align*}$$

Now taking a sequence of increasing compact sets $\{K_n\}$ such that $\cup_{n \in \mathbb{N}} K_n = \Omega$ and a sequence $\{\varepsilon_1^n\}_{n \in \mathbb{N}}$ and $\{\varepsilon_2^n\}_{n \in \mathbb{N}}$ such that $\varepsilon_i^n \to 0$ as $n \to +\infty$ for $i = 1, 2$, using a diagonal extraction process and estimates (23) and (24), there exist subsequences still denoted $\{\varepsilon_1^n\}_{n \in \mathbb{N}}$ and $\{\varepsilon_2^n\}_{n \in \mathbb{N}}$ such that as $n \to +\infty$

$$\begin{align*}
u_{\varepsilon_1^n} & \to u \text{ pointwise in } \Omega 
\end{align*}$$

and

$$\begin{align*}
u_{\varepsilon_2^n} & \to v \text{ pointwise in } \Omega.
\end{align*}$$

From ((23) and (24), $(u, v)$ satisfies (6). Then, from classical regularity theory, it is easy to show that $(u, v) \in C^{1,\delta}_{\text{loc}}(\Omega)^2$ for some $0 < \delta < 1$ and is a weak solution to (1). This completes the proof of Proposition 8.

\section{Uniqueness results}

Let us assume first that (2) and (3) hold. From Theorem B.1 in [15], we have that every weak solution satisfying (5) belongs to $C^{1,\delta}(\Omega)^2$ for some $0 < \delta < 1$. To prove the uniqueness, we will use the following strong comparison principle which extends Theorem 2.3 in [15]:

**Proposition 9** Let $u_1, u_2 \in C^{1,\delta}(\overline{\Omega})$ for some $0 < \delta < 1$, satisfy $u_1, u_2 > 0$ and

$$\begin{align*}
-\Delta_p u_1 - h. u_1^{-\beta} = f, \\
-\Delta_p u_2 - h. u_2^{-\beta} = g,
\end{align*}$$

with $u_1 = u_2 = 0$ on $\partial \Omega$, where $f, g \in C(\Omega)$ are such that $0 \leq f < g$ pointwise everywhere in $\Omega$ and $h \in C^{0,\delta}(\overline{\Omega})$ such that $0 \leq h(x)d(x)^{-\beta} \leq d(x)^\alpha$ for some $\alpha > -1$. Then, the following strong comparison principle holds:

$$\begin{align*}
0 < u_1 < u_2 \text{ in } \Omega \text{ and } \frac{\partial u_2}{\partial \nu} < \frac{\partial u_1}{\partial \nu} < 0 \text{ on } \partial \Omega.
\end{align*}$$

We are ready to prove the following uniqueness result:

**Theorem 10** Let (2) and (3) hold. Then there exists a unique weak solution $(u, v)$ satisfying (5). Moreover $(u, v) \in C^{1,\delta}(\Omega)^2$ for some $0 < \delta < 1$.

**Proof** Let us assume by contradiction that there exist two distinct solutions $(u, v)$ and $(\tilde{u}, \tilde{v})$ satisfying both (5). Then we adapt a method due to Krasnoleskii :

$$\begin{align*}
\epsilon_{\text{max}} = \sup\{\epsilon \in [0, 1] \mid \epsilon u \leq \tilde{u} \text{ and } \epsilon v \leq \tilde{v} \text{ in } \Omega\}
\end{align*}$$

\[\epsilon_{\text{max}} = \sup\{\epsilon \in [0, 1] \mid \epsilon u \leq \tilde{u} \text{ and } \epsilon v \leq \tilde{v} \text{ in } \Omega\} \]
Since \((u, v)\) and \((\tilde{u}, \tilde{v})\) satisfy (5), \(u, v, \tilde{u}, \tilde{v}\) belong to \(C^{1,\delta}(\Omega)\) for some \(0 < \delta < 1\). Therefore, \(0 < c_{\text{max}} \leq 1\). If \(c_{\text{max}} = 1\) and since the roles of \((u, v)\) and \((\tilde{u}, \tilde{v})\) are interchangeable, the proof of Theorem 10 is complete. Hence assume that \(c_{\text{max}} < 1\). Then,
\[
-\Delta_{p_1}(c_{\text{max}}u) = c_{\text{max}}^{p_1 - 1}u^{\alpha_1}v^{\beta_1} < (c_{\text{max}}u)^{\alpha_1}v^{\beta_1} \quad \text{in} \quad \Omega,
\]
and
\[
-\Delta_{p_2}(c_{\text{max}}v) = c_{\text{max}}^{p_2 - 1}u^{\alpha_2}v^{\beta_2} < (c_{\text{max}}v)^{\alpha_2}v^{\beta_2} \quad \text{in} \quad \Omega.
\]
From (30), (31) and Proposition 9, we get that
\[
c_{\text{max}}u < \tilde{u}, \quad c_{\text{max}}v < \tilde{v} \quad \text{in} \quad \Omega,
\]
and
\[
0 > \frac{\partial(c_{\text{max}}u)}{\partial \nu} > \frac{\partial \tilde{u}}{\partial \nu} \quad \text{on} \quad \partial \Omega.
\]
From (20), (33) and (34), we get a contradiction with the definition of \(c_{\text{max}}\). Then, \(c_{\text{max}} = 1\). □

6 Proofs of Main results

PROOF [Proof of Theorem 1] It follows from Proposition 7. □

PROOF [Proof of Theorem 2] It follows from Theorem B.1. in [15] and Theorem 10. □

PROOF [Proof of Theorem 3] It follows from Proposition 8. □

PROOF [Proof of Theorem 4] From Hardy Inequality, \(d(x)^{-\delta}\) belongs to \(W^{-1,\frac{p}{p-1}}(\Omega)\), if and only if \(\delta < 2 - \frac{1}{p}\). Therefore, \((u, v)\), a weak solution to (1), satisfying (6) belongs to \(W^{1,p_1}_0(\Omega) \times W^{1,p_2}_0(\Omega)\) if and only if (8) holds. □

7 Appendix

We give a \(L^\infty\)-bound to weak solutions in \(W^{1,p_1}_0(\Omega) \times W^{1,p_2}_0(\Omega)\) to (1) by using Moser iterations:

Lemma 11 Let \((u, v) \in W^{1,p_1}_0(\Omega) \times W^{1,p_2}_0(\Omega)\) be a weak solution of (1). Then \((u, v) \in (L^\infty(\Omega))^2\).

PROOF For \(M > 0\) define
\[
\begin{aligned}
\{ \quad u_M(x) &= \min(u_n(x), M) \\
v_M(x) &= \min(v_n(x), M)
\end{aligned}
\]
and set
\[
(\varphi, \psi) = \left(u^{kp_1+1}_M, v^{kp_2+1}_M\right)
\]
with \(k > 0\) in (9). Then
\[
(kp_1 + 1) \int_\Omega |\nabla u_M|^{p_1} u^{kp_1}_M = \int_\Omega u^{\alpha_1}v^{\beta_1}u^{kp_1+1}_M
\]
(35)
and
\[(kp_2 + 1) \int_{\Omega} |\nabla v_M|^{p_2} v_M^{kp_2} = \int_{\Omega} u^{\alpha_2} v^{\beta_2} v_M^{kp_2}.\] (36)

The left-hand side \(L_1\) and \(L_2\) of (35) and (36), respectively, are estimated from below by

\[L_1 = (kp_1 + 1) \int_{\Omega} |\nabla u_M|^{p_1} u_M^{kp_1} = \frac{(kp_1 + 1)}{(k+1)p_1} \int_{\Omega} (\nabla u_M^{k+1})^{p_1} \geq C_1 \left( \int_{\Omega} u_M^{k+1} \right)^{p_1} \] (37)

and

\[L_2 = (kp_2 + 1) \int_{\Omega} |\nabla v_M|^{p_2} v_M^{kp_2} = \frac{(kp_2 + 1)}{(k+1)p_2} \int_{\Omega} (\nabla v_M^{k+1})^{p_2} \] (38)

We have \(\alpha_i < p_i - 1\) \((i = 1, 2)\) then \(kp_i + \alpha_i + 1 < p_i (k + 1)\). Thus, the right-hand side \(R_1\) and \(R_2\) of (35) and (36), respectively, are estimated from above by

\[R_1 = \int_{\Omega} u^{kp_1+\alpha_1+1} v^{\beta_1} \leq \left( \int_{\Omega} u^{(k+1)p_1} \right)^{\frac{kp_1+\alpha_1+1}{(k+1)p_1}} \left( \int_{\Omega} v^{\beta_1+\alpha_1+1} \right)^{\frac{\beta_1+\alpha_1+1}{(k+1)p_1}} \]

\[ \leq C_2 \|v\|^\beta_1 \left( \int_{\Omega} u^{(k+1)p_1} \right)^{\frac{p_1}{\alpha_1}} \]

\[ \leq C_3 \left( \int_{\Omega} u^{q_1(k+1)} \right)^{\frac{p_1}{\alpha_1}} \]

and

\[R_2 = \int_{\Omega} u^{\alpha_2} v^{kp_2+\beta_2+1} \leq C_4 \left( \int_{\Omega} v^{q_2(k+1)} \right)^{\frac{p_2}{\beta_2}}, \]

where \(C_i\) \((i = 2, 4)\) are positive constants. Hence from (37) and (38) we get

\[
\begin{cases}
\left( \int_{\Omega} u_M^{k+1} \right)^{\frac{p_1}{p_1}} \leq C_4 \left( \int_{\Omega} v^{q_2(k+1)} \right)^{\frac{p_2}{p_2}} \\
\left( \int_{\Omega} v_M^{k+1} \right)^{\frac{p_2}{p_2}} \leq C_4 \left( \int_{\Omega} u^{q_1(k+1)} \right)^{\frac{p_1}{p_1}}
\end{cases}
\]

i.e.

\[
\begin{cases}
\|u_M\|_{(k+1)p_1} \leq C_5 \left( \frac{k+1}{(kp_1+1)^{\frac{1}{p_1}}} \right)^{\frac{1}{k+1}} \|u\|_{(k+1)q_1} \\
\|v_M\|_{(k+1)p_2} \leq C_5 \left( \frac{k+1}{(kp_2+1)^{\frac{1}{p_2}}} \right)^{\frac{1}{k+1}} \|v\|_{(k+1)q_2}
\end{cases}
\] (39)

Choosing \(k_1\) and \(\bar{k}_1\) in (39) such that

\[(k_1 + 1) q_1 = p_1^* \text{ and } (\bar{k}_1 + 1) q_2 = p_2^*.\]
i.e.

\[ k_1 = \left( \frac{p_1^*}{q_1} \right) - 1 \quad \text{and} \quad \overline{k}_1 = \left( \frac{p_2^*}{q_2} \right) - 1. \]

Then (39) (with \( k_1 \) and \( \overline{k}_1 \)) holds for any \( M > 0 \) and we can start Moser iterations in (39) (see [10], pp. 112, 113) to get sequences \( (r_m, \overline{r}_m) \to \infty \) and constant \( c_6, c_6^* > 0 \) independent of \( m \) such that

\[ \| u \|_{L^{r_m}(\Omega)} \leq c_6 \| u \|_{L^{p_1^*}(\Omega)} < \infty \]

and

\[ \| v \|_{L^{\overline{r}_m}(\Omega)} \leq c_6^* \| v \|_{L^{p_2^*}(\Omega)} < \infty. \]

Hence

\[ \| u \|_{L^{\infty}(\Omega)} \leq c_6 \| u \|_{L^{p_1^*}(\Omega)} < \infty \]

and

\[ \| v \|_{L^{\infty}(\Omega)} \leq c_6^* \| v \|_{L^{p_2^*}(\Omega)} < \infty, \]

which complete the proof of lemma.

\[ \square \]

References


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