The intersection graph of gamma sets in the total graph of a commutative ring II

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Abstract
T. Tamizh Chelvam and T. Asir [15] studied the intersection graph of gamma sets in the total graph of a commutative Artin ring. The intersection graph $I_T\Gamma(R)$ of gamma sets in the total graph of a commutative ring $R$, is the undirected graph with vertex as collection of all $\gamma$-sets in the total graph of $R$ and two distinct vertices $u$ and $v$ are adjacent if and only if $u \cap v \neq \emptyset$. In this paper, we continue our interest on $I_T\Gamma(R)$ and actually we study about Eulerian and Hamiltonian nature of $I_T\Gamma(R)$. Further, we focus on certain graph theoretic parameters of $I_T\Gamma(R)$ like the independence number and the clique number of $I_T\Gamma(R)$. Some of the results proved in this paper generalize the results proved in [13]. Keywords: Artin ring, total graph, intersection graph, Hamiltonian, independence number.

1 Introduction
The present paper is a sequel to [15] and so the notations introduced in the introduction of [15] will remain in force. Thus throughout this paper, $R$ denotes a commutative ring with identity $1 \neq 0$, $2 = 1 + 1$ and $Z(R)$ denotes the the set of all zero divisors of $R$. For any $a \in R$, $Ann(a) = \{x \in R : ax = 0\}$ is the annihilator ideal of $a$ in $R$ and the ideal generated by $a$ is denoted by $(a)$. Throughout this paper, let $I$ be an annihilator ideal with $|R/I|$ is finite and $|R/I| = \min\{|R/J| : J$ is an annihilator ideal of $R\}$. Let us take $|I| = \lambda$, $|Z(R)| = \alpha$, $|R/Z(R)| = \beta$ and $|R/I| = \mu$. For a general reference on rings, we refer to Kaplansky [7].

Anderson and Badawi [2] introduced the concept of the total graph corresponding to a commutative ring. The total graph of $R$, denoted by $T_T(R)$, is the undirected graph with vertices $R$, and for distinct $x, y \in R$, the vertices $x$ and $y$ are adjacent if $x + y \in Z(R)$. There after various research articles have been published on the total graph of a commutative ring [1, 9, 11, 12, 14]. The intersection graph of gamma sets in the total graph of a commutative ring $R$ is the undirected graph with vertex as collection of all $\gamma$-sets in the total graph of $R$ and two distinct vertices $u$ and $v$ are adjacent if and only if $u \cap v \neq \emptyset$. Now we continue the investigation of interplay between some graph-theoretic properties of $I_T(R)$ and the ring-theoretic properties of $R$.

Let $G = (V, E)$ be a finite graph with vertex set $V$ and edge set $E$. A subset $S$ of $V$ is called a dominating set in $G$ if every vertex in $V - S$ is adjacent to at least one vertex in $S$. The domination number of $G$, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set in $G$ and such a dominating set is called $\gamma$-set of $G$. A set of vertices in $G$ is said to be independent if no two vertices in the set are adjacent. The independence number $\beta_0(G)$, is the maximum cardinality of an independent set in $G$. The clique number $\omega(G)$, is the number of vertices in a

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largest complete subgraph of $G$. Let $G_1$ and $G_2$ be two graphs. The union of $G_1$ and $G_2$, denoted by $G_1 \cup G_2$, is a graph with vertex set $V(G_1) \cup V(G_2)$ and edge-set $E(G_1) \cup E(G_2)$. For graph theory parameters, we refer reader to [6].

**Theorem 1** [15, Theorem 2.4] Let $R$ be a commutative Artin ring which is not an integral domain. Assume that $I_i$‘s ($i = 1, 2, \ldots, t$) are the annihilator ideals of $R$ such that $|R/I_i| = \mu$ is finite and minimum, and $R/I_i = \{a_{ij} + I_i : 1 \leq j \leq \mu\}$ for each $i$, $1 \leq i \leq t$. Then $\{x_1, x_2, \ldots, x_\mu\}$ is a $\gamma$-set of $\Gamma(R)$ if and only if $x_j \in a_{ij} + I_i$ for $j = 1, \ldots, \mu$ and for some fixed $i$.

**Lemma 2** [15, Lemma 3.3] Let $R$ be a finite commutative ring and $I_i$ ($i = 1, 2, \ldots, t$) be annihilator ideals in $R$ such that $|R/I_i|$ is minimum. Let $|I_i| = \lambda$, $|R/I_i| = \mu$ and $k = |\{x \in R : 2x = 0\}|$. Then

(i) 
$$|V(\Gamma_{TT}(R))| = \begin{cases} 
2^{\frac{|R| - k}{2}} & \text{if } R \text{ is an integral domain} \\
\lambda^\mu & \text{if } t = 1 \\
2\mu^\mu - \mu!(\frac{\lambda}{\mu})^\mu & \text{if } t = 2.
\end{cases}$$

Moreover if $t \geq 3$, then $|V(\Gamma_{TT}(R))| \geq 2\mu^\mu - \mu!(\frac{\lambda}{\mu})^\mu > \lambda^\mu$.

(ii) If $t = 1$, then $\deg(v) = \sum_{i=1}^{\mu} [\lambda^{\mu-i}(\lambda - 1)^{i-1}] - 1$ for all $v \in V(\Gamma_{TT}(R))$. Further if $\mu = 2$, then $\deg(v) = |R| - 2$ for all $v$.

(iii) If $t \geq 2$, then $\deg(v) \geq \sum_{i=1}^{\mu} [\lambda^{\mu-i}(\lambda - 1)^{i-1}] - 1 + (\frac{\lambda}{\mu})\lambda^{\mu-2}$ for all $v \in V(\Gamma_{TT}(R))$.

2 Eulerian and Hamiltonian nature of $\Gamma_{TT}(R)$

In this section, we are interested in the Eulerian and Hamiltonian nature of $\Gamma_{TT}(R)$. Throughout this section, we assume that $|R| \geq 4$ in order to avoid $|V(\Gamma_{TT}(R))| \leq 3$. We begin this section with a result concerning Eulerian nature of $\Gamma_{TT}(R)$.

**Theorem 3** Let $R$ be a finite commutative ring with $|R| \geq 4$ and $I$ be the unique annihilator ideal in $R$ such that $|R/I|$ is minimum. Then $\Gamma_{TT}(R)$ is Eulerian if and only if $R$ is not a field.

**Proof** If $R$ is a field, then $\Gamma_{TT}(R)$ is a complete graph with even number of vertices and so is not Eulerian. Assume that $R$ is not a field. Let $|I| = \lambda$ and $|R/I| = \mu$. If $\lambda$ is odd, then $\lambda^{\mu-1}$ is odd and so by Lemma 2(ii), $\deg(v)$ is even for all $v \in V(\Gamma_{TT}(R))$. If $\lambda$ is even, then $(\lambda - 1)^{\mu-2}$ is odd and so by Lemma 2(ii), $\deg(v)$ is even for all $v \in V(\Gamma_{TT}(R))$. Hence $\Gamma_{TT}(R)$ is Eulerian. \[\Box\]

In 1969, Lovász[8] posed that whether every finite connected vertex-transitive graph has a Hamilton path. However, the general problem of finding Hamilton paths and cycles in highly symmetric graphs may be much older. After all these years, a connected vertex-transitive graph without a Hamilton path is yet to be produced. Moreover, only four connected vertex-transitive graphs (having at least three vertices) not having a Hamilton cycle are known to exist: the Petersen graph, the Coxeter graph, and the two graphs obtained from them by replacing each vertex with a triangle. Particular attention has been given to Cayley graphs. For example, one may easily see that connected Cayley graphs of abelian groups have a Hamilton cycle. As seen in Theorem 4.1 [15], $\Gamma_{TT}(R)$ is a vertex-transitive graph if $R$ contains only one annihilator ideal.
Let \( |R/I| \) is finite and minimum. Further \( I_{TT}(R) \) is a union of vertex-transitive subgraphs of \( I_{TT}(R) \). Thus a natural question is whether \( I_{TT}(R) \) is Hamiltonian or not. In this regard, we prove that \( I_{TT}(R) \) is Hamiltonian and so we get a class of vertex-transitive Hamiltonian graphs. Hence a subclass of \( I_{TT}(R) \) for various commutative rings \( R \), satisfies Lovász conjecture.

In order to prove that \( I_{TT}(R) \) is Hamiltonian, we start with a lemma which is used frequently.

**Lemma 4** Let \( R \) be a commutative Artin ring which is not an integral domain and \( I_i \) \((i = 1, \ldots, t)\) be the annihilator ideals in \( R \) such that \( |R/I_i| = \mu \) is finite and minimum. Let \( \bigcup_{i=1}^{t} I_i = \{a_1, \ldots, a_r\}, A_1 = \{v \in V(I_{TT}(R)) : a_1 \in v\} \) and \( A_k = \{v \in V(I_{TT}(R)) : a_k \in v \text{ and } v \notin \bigcup_{m=1}^{k-1} A_m\} \) for \( 2 \leq k \leq r \). Then \( A_k \) is complete and \( A_k \cup A_{\ell} \) is connected as subgraphs of \( I_{TT}(R) \) for \( k \neq \ell, 1 \leq k, \ell \leq r \).

**Proof** Since \( R \) is Artin, \( R \cong R_1 \times \cdots \times R_n \) where each \( R_p \) is a local ring with \( Z(R_p) = m_p \) for \( p = 1, \ldots, n \). Note that every maximal annihilator ideal in \( R \) is of the form \( R_1 \times \cdots \times R_{p-1} \times m_p \times R_{p+1} \times \cdots \times R_n \) for \( p = 1, \ldots, n \). In view of this \( I_i = R_1 \times \cdots \times R_{i-1} \times m_i \times R_{i+1} \times \cdots \times R_n \) for \( i = 1, \ldots, t \). Since \( |R/I_i| = \mu, |R_i/m_i| = \mu \), and so let \( R_i/m_i = \{x_{1i} + m_i = m_i, x_{2i} + m_i, \ldots, x_{mi} + m_i\} \) with \( x_{ji} \in R_i \) for \( j = 1, \ldots, m_i \) and \( i = 1, \ldots, t \). Let \( y_{ij} = (b_1, b_2, \ldots, b_{i-1}, x_{ij}, b_{i+1}, \ldots, b_n) \in R_1 \times \cdots \times R_n \) where \( b_p \in R_p \). Then \( R/I_i = \{y_{ij} + I_i : 1 \leq j \leq \mu\} \) for \( i = 1, \ldots, t \). By the choice \( x_{1i} \in m_i, y_{i1} \in I_i \) for \( i = 1, \ldots, t \). Since \( a_k \in v \) for every \( v \in A_k, < A_k > \subseteq I_{TT}(R) \) is complete.

To prove the other part, first we show that for each \( c \in \bigcup_{i=1}^{k-1} A_m \) and so \( u \not\in A_k \). Since \( z_\mu \) is an arbitrary, for each \( c \in \bigcup_{i=1}^{k-1} A_m \) and so \( u \not\in A_k \). Since \( z_\mu \) is an arbitrary, for each \( c \in \bigcup_{i=1}^{k-1} A_m \) and so \( u \not\in A_k \). Since \( z_\mu \) is an arbitrary, for each \( c \in \bigcup_{i=1}^{k-1} A_m \) and so \( u \not\in A_k \). Since \( z_\mu \) is an arbitrary, for each \( c \in \bigcup_{i=1}^{k-1} A_m \) and so \( u \not\in A_k \). Since \( z_\mu \) is an arbitrary, for each \( c \in \bigcup_{i=1}^{k-1} A_m \) and so \( u \not\in A_k \). Since \( z_\mu \) is an arbitrary, for each \( c \in \bigcup_{i=1}^{k-1} A_m \) and so \( u \not\in A_k \).

The following theorem concerns about the existence of a Hamiltonian cycle in \( I_{TT}(R) \).

**Theorem 5** Let \( R \) be a commutative Artin ring with \( |R| \geq 4 \) and \( I \) be an annihilator ideal of \( R \) such that \( |R/I| \) is minimum. Then \( I_{TT}(R) \) is Hamiltonian.

**Proof** When \( R \) is an integral domain, \( I_{TT}(R) \) is complete and so trivially Hamiltonian. Let \( R \) be a commutative Artin ring which is not an integral domain. As per the notations of Lemma 4, \( A_k \subseteq I_{TT}(R) \) is complete and \( A_k \cup A_{\ell} \) is connected for all \( k \neq \ell \).

Let \( B = (x_{12} + m_1) \times (x_{22} + m_2) \times \cdots \times (x_{j2} + m_j) \times R_{12} \times \cdots \times R_{n1} \).

Suppose \( |B| \geq 3 \) and let \( c_1, c_2, c_3 \in B \). By proof of Lemma 4, for each \( k = 1, \ldots, r \), there exists \( \{u_k, v_k, w_k\} \subseteq A_k \) such that \( c_1 \in u_k, c_2 \in v_k \) and \( c_3 \in w_k \). Now start with the vertex \( u_1 \in A_1 \) and pass on to a vertex \( u_2 \in A_2 \), traverse the vertices in \( < A_2 > \) through a spanning path in \( < A_2 > \) ending at \( v_2 \in A_2 \). Then pass on to \( < A_3 >, < A_4 >, < A_5 >, \ldots, < A_{r-1} > \) to get a Hamiltonian path ending at \( v_{r-1} \in A_{r-1} \) or \( v_{r-1} \in A_{r-1} \), say \( u_{r-1} \in A_{r-1} \). Now pass on...
to the vertex $u_r \in A_r$, traverse vertices in $< A_r >$ through a spanning path in $< A_r >$ ending at $w_r \in A_r$. Then pass on to the vertex $w_1 \in A_1$, traverse vertices in $< A_1 >$ through a spanning path in $< A_1 >$ and ending at $u_1$ gives a required Hamiltonian cycle in $I_{TT}(R)$.

Suppose $|B| \leq 2$. If $R$ is local, then $|x_{\mu} + Z(R)| = 2$ and so $R \cong \mathbb{Z}_2$ or $\mathbb{Z}_2[x]/(x^2)$. In this case $I_{TT}(R) = C_4$. If $R$ is not local, then $|x_{\mu_i} + m_i| = 1$ for $i = 1, 2, \ldots, t$, and so $R \cong \mathbb{Z}_2 \times \ldots \times \mathbb{Z}_2$. Now, for each element $y \in R$, there is $v \in A_1$ such that $y \in v$ and $A_r$ is a singleton set. Thus as discussed above one can get a Hamiltonian cycle in $I_{TT}(R)$.

\section{Independent and Clique numbers of $I_{TT}(R)$}

In this section, we obtain the values of independence and clique numbers of $I_{TT}(R)$. First we find the vertex and edge independence numbers of $I_{TT}(R)$.

**Lemma 6** Let $R$ be a finite commutative ring, $I$ be an annihilator ideal in $R$ with $|R/I| = \mu$ is minimum and $|I| = \lambda$. Then the independence number $\beta_0(I_{TT}(R)) = \lambda$.

**Proof** Let $I = \{a_1, \ldots, a_{\lambda}\}$ and $R/I = \{I, x_1+I, \ldots, x_{\mu-1}+I\}$. Let $u_i = \{a_i, x_1+a_i, \ldots, x_{\mu-1}+a_i\}$ for $i = 1, \ldots, \lambda$ and $S = \{u_1, u_2, \ldots, u_{\lambda}\}$. Then $S$ is an independence set in $I_{TT}(R)$ and subsets in $S$ is a partition of $R$. Hence $S$ is a maximum independent set in $I_{TT}(R)$.

\hfill $\Box$

**Lemma 7** Let $R$ be a finite commutative ring. Then the edge independence number $\alpha'(I_{TT}(R)) = \left\lceil \frac{|V(I_{TT}(R))|}{2} \right\rceil$. In particular $I_{TT}(R)$ has a perfect matching if and only if $|V(I_{TT}(R))|$ is even.

**Proof** If $R$ is an integral domain, then $I_{TT}(R)$ is a complete graph with even number of vertices and so $\alpha'(I_{TT}(R)) = \frac{|V(I_{TT}(R))|}{2}$. Assume that $R$ is not an integral domain. Let $I_i$’s be annihilator ideals in $R$ such that $|R/I_i|$ is minimum for $i = 1, \ldots, t$ and $\bigcup_{i=1}^{t} I_i = \{a_1, \ldots, a_r\}$.

Let $A_1 = \{v \in V(I_{TT}(R)) : a_1 \in v\}$ and $A_j = \{v \in V(I_{TT}(R)) : a_k \in v$ and $v \notin \bigcup_{m=1}^{k-1} A_m\}$ for $2 \leq j \leq r$.

Suppose $|V(I_{TT}(R))|$ is even. If $|A_k|$ is even for all $k = 1, \ldots, r$, then $< A_i >$ has a perfect matching and hence $I_{TT}(R)$ has a perfect matching. If $|A_k|$ and $|A_\ell|$ are odd for some $k, \ell$ with $1 \leq k \neq \ell \leq r$. By Lemma 4, $< A_k \cup A_\ell >$ is connected for $k \neq \ell$ and so there exists $u \in A_k$ and $v \in A_\ell$ such that $uv \in E(I_{TT}(R))$. Consider a maximum matching $M_k$ of $< A_k >$ not containing $u$ and a maximum matching $M_\ell$ of $< A_\ell >$ not containing $v$, then $M_k \cup M_\ell \cup \{uv\}$ is a perfect matching of $< A_k \cup A_\ell >$. Proceeding in this way, one can get a perfect matching of $I_{TT}(R)$ and so $\alpha'(I_{TT}(R)) = \frac{|V(I_{TT}(R))|}{2}$. If $|V(I_{TT}(R))|$ is odd, then as proved above, we have $\alpha'(I_{TT}(R)) = \frac{|V(I_{TT}(R))|-1}{2}$.

\hfill $\Box$

**Theorem 8** Let $R$ be a finite commutative ring and $I_i$ $(i = 1, \ldots, t)$ be the annihilator ideals in $R$ such that $|R/I_i| = \mu$ is minimum and $|I_i| = \lambda$. Then the clique number $\omega(I_{TT}(R)) = \frac{|V(I_{TT}(R))|}{\lambda}$. In particular, if $k = |\{x \in R : 2x = 0\}|$, then

$$
\omega(I_{TT}(R)) = \begin{cases} 
2^{\frac{|m-k|}{2}} & \text{if } R \text{ is an integral domain} \\
\lambda^{t-1} & \text{if } t = 1 \\
2\lambda^{t-1} - (\mu - 1)! (\frac{1}{\mu})^{t-1} & \text{if } t = 2
\end{cases}
$$
and $\omega(I_{TT}(R)) > \lambda^{\mu-1}$ if $t \geq 3$.

**Proof** If $R$ is an integral domain, then $I_{TT}(R)$ is complete and so $\omega(I_{TT}(R)) = |V(I_{TT}(R))| = 2^{\frac{|R|-1}{2}}$. Assume that $R$ is not an integral domain. Let $\bigcup_{i=1}^{t} I_i = \{a_1, \ldots, a_t \}$, $A_1 = \{v \in V(I_{TT}(R)) : a_1 \in v \}$ and $A_j = \{v \in V(I_{TT}(R)) : a_k \in v \text{ and } v \notin \bigcup_{m=1}^{k-1} A_m \}$ for $2 \leq j \leq r$.

Note that $<A_k>$ is complete and $|A_1| \geq |A_j|$ for all $2 \leq j \leq r$.

We claim that no vertex in $V(I_{TT}(R)) - A_1$ is adjacent to all vertices in $A_1$. For, let $u = \{x_1, \ldots, x_\mu \} \in V(I_{TT}(R)) - A_1$ and so $x_i \neq a_1$ for all $i$. Suppose $u$ is a $\gamma$-set of $T_\Gamma(R)$ with respect to $I_p$ for some $p$, $1 \leq p \leq t$. Assume that $a_1 \in y + I_p$. By Theorem 1, there exists some $j(1 \leq j \leq \mu)$ such that $x_j \in y + I_p$. Since $x_i \neq a_1$ for all $i$, $0 \neq z = a_1 - x_j \in I_p$. From this $w = \{x_1 + z, \ldots, x_{j-1} + z, a_1, x_{j+1} + z, \ldots, x_\mu + z \} \in A_1$ and $w$ is not adjacent to $u$. Similarly one can prove that for any element $x \in R$ and $S = \{v \in V(I_{TT}(R)) : x \in v \}$, $<S>$ is a maximal clique. Also one can note that any maximal clique will be of this form. Thus $\omega(I_{TT}(R)) = |A_1|$.

To obtain, the value of $|A_1|$, let $v_1 = \{a_1, x_2, \ldots, x_\mu \}$ be an arbitrary vertex in $A_1$. By Theorem 1, there exists a least positive integer $i(1 \leq i \leq t)$ such that $v_1$ is a $\gamma$-set of $T_\Gamma(R)$ with respect to $I_i$. Let $I_i = \{0, b_2, b_3, \ldots, b_\lambda \}$ and $v_j = \{a_1 + b_j, x_2 + b_j, \ldots, x_\mu + b_j \}$ for $j = 2, 3, \ldots, \lambda$. From this construction, corresponding to each arbitrary vertex in $A_1$, one can associate $\lambda - 1$ vertices in $V(I_{TT}(R)) - A_1$. This gives a partition for $V(I_{TT}(R))$ into $|A_1|$ subsets each containing $\lambda$ elements. Hence $\omega(I_{TT}(R)) = |A_1| = \frac{|V(I_{TT}(R))|}{\lambda}$. Particular cases from follow from Lemma 2. □

Let $G$ be a graph. For any $\ell$ with $1 \leq \ell \leq \left\lfloor \frac{|V(G)|}{2} \right\rfloor$, $K_{\ell, \ell}$ is a subgraph of $G$. Further for some $\ell > \left\lfloor \frac{|V(G)|}{2} \right\rfloor$, $K_{\ell, \ell}$ may be a subgraph of $G$. Now we identify an upper bound for such a number $\ell$, whenever $\mu = 2$.

**Theorem 9** Let $R$ be a finite commutative ring, $|R| > 2$ and $I$ be an annihilator ideal of $R$ with $|R/I| = 2$. Then the following are true:

(i) If $R$ is either $\mathbb{Z}_4$ or $\frac{\mathbb{Z}_4[x]}{x^2}$ or $\mathbb{Z}_2 \times \mathbb{Z}_2$ or $\mathbb{Z}_6$, then $K_{\ell, \ell}$ is a subgraph of $I_T(R)$ if and only if either $\ell = 1$ or 2.

(ii) If $R$ is not isomorphic to the rings in (i), then $K_{\ell, \ell}$ is a subgraph of $I_{TT}(R)$ if and only if $1 \leq \ell \leq \left\lfloor \frac{\omega(I_{TT}(R))}{2} \right\rfloor$.

**Proof** (i) If $R$ is either $\mathbb{Z}_4$ or $\frac{\mathbb{Z}_4[x]}{x^2}$ or $\mathbb{Z}_2 \times \mathbb{Z}_2$, then by Example 3.1 [15], $\ell \leq 2$. If $R = \mathbb{Z}_6$, then the subgraph induced by vertices $\{0, 1\}, \{2, 3\}, \{0, 3\}$ and $\{1, 2\}$ is a maximal complete bi-partite graph of $I_{TT}(R)$ and so $\ell \leq 2$.

(ii) Let $|I| = \lambda$ and $|R/I| = \mu = 2$. If $R$ is not isomorphic to the rings in (i), then $\lambda \geq 4$ and so by Theorem 8, $\omega(I_{TT}(R)) \geq \lambda^{\mu-1} \geq 4$.

Let $s = \left\lfloor \frac{\omega(I_{TT}(R))}{2} \right\rfloor + 1$ if $\omega(I_{TT}(R))$ is even

otherwise.

Suppose $K_{s,s}$ is a subgraph of $I_T(R)$ with vertex partition $(X, Y)$. Let $X = \{u_1, \ldots, u_s \}$ and $Y = \{v_1, \ldots, v_s \}$. Since $\omega(I_{TT}(R)) \geq 4$, we get that $s \geq 3$. Note that each vertex in $X$ as well as $Y$ is a subset of $R$ with two elements.

Suppose there is some $x \in u_1 \cap \ldots \cap u_s$. Since $|X| \geq 3$ and $|Y| \geq 3$, $x \in v_j \in Y$ for all $j$ with $1 \leq j \leq s$. Hence $K_{2s}$ is a subgraph of $I_{TT}(R)$ and so $\omega(I_{TT}(R)) \geq 2s$, a contradiction to $\omega(I_{TT}(R)) < 2s$. 


Suppose there is some $x \in R$ and $x$ is in some subsets in $X$, and $y \in R$ is in some other subsets in $X$. Now, there is at most only one vertex $\{x, y\} \in V(I_{TT}(R))$ which is adjacent to all the vertices $X$. That is, $\{x, y\}$ is the only vertex in $Y$, a contradiction to $|Y| \geq 3$. Therefore $K_{s,s}$ is not a subgraph of $I_{T}(R)$. □

Having obtained the upper bound for $\ell$ when $\mu = 2$, we propose the following open problem when $\mu \geq 3$.

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**References**


