Numerical methods for highly oscillatory integrals on semi-finite intervals

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Abstract

In highly oscillatory integrals, the integrand \( f_{\omega}(x) \) oscillates rapidly with a frequency \( \omega \). For very high values of \( \omega \), numerical evaluation of such integrals by Gaussian quadrature rules can be of very low accuracy. In such problems which have many applications in mathematical physics, it is important to devise algorithms with errors which decrease as fast as \( \omega^{-N} \), for some \( N > 0 \). In this paper, we review some existing methods, and particularly, Levin-type methods. Then we present a steepest descent method which converts highly oscillatory integrals to non-oscillatory integrals. The idea behind this method is to replace the integration interval with a path in the complex plane, in such a way that the oscillatory factor of integrand decays exponentially along this path. The method is a modification of a 2006-SIAM paper for highly oscillatory integrals on bounded intervals. **Keywords:** highly oscillatory integrals, numerical steepest descent, unbounded intervals.

1 Introduction

In this paper we consider numerical approximation of the integral

\[
I := \int_0^\infty f(x)e^{i\omega g(x)} \, dx,
\]

where \( f \in L^1([0, +\infty)) \), \( g \) is, in general, a piecewise smooth function on \([0, +\infty)\), and \( \omega > 0 \) is a constant. Direct application of interpolatory quadrature rules (even higher order ones, e.g., Gaussian rules), when the frequency parameter \( \omega \) is large, results in very poor accuracies. Indeed, the accuracy deteriorates as \( \omega \) decreases. Thus it is important to design numerical methods with errors that decreases for increasing \( \omega \). Researchers specially are interested in numerical methods with errors that scales as \( \mathcal{O}(\omega^{-N}) \) for some \( N > 0 \). Some successful methods are proposed in the last two decades, among which the most important ones are the Filon method [1], Levin collocation method [3], and numerical steepest decent [2].

In this paper, we focus on the numerical steepest decent of [2] and extend it to the unbounded interval \([0, +\infty)\). This is done by careful selection of the path of steepest decent in the complex plane. This work can be regarded as a brief report on the manuscript [4].

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2 Existing methods

In this section, we briefly review three main approaches for approximation of highly oscillatory integrals of the form (1) (on bounded and/or unbounded intervals).
2.1 Filon method

Consider the integral (1) with bounded domain $\mathcal{I} := [a, b]$. In Filon method, as introduced for the first time in [1], $\mathcal{I}$ is divided into $2n$ subintervals

$$\mathcal{I}_k := [a + kh, a + (k + 1)h], \quad h := \frac{b - a}{2n}, \quad k = 0, \ldots, 2n - 1,$$

for a large enough integer $n > 0$. Then for each $k = 0, \ldots, n - 1$, the factor $f$ is approximated by a quadratic $p_{2,k}(x)$ via polynomial interpolation at three adjacent points $a + 2kh$, $a + (2k + 1)h$, and $a + (2k + 2)h$. As the result, the following approximation is obtained.

$$I \approx \sum_{k=0}^{n-1} \int_{a+2kh}^{a+(2k+2)h} p_{2,k}(x) e^{i\omega g(x)} \, dx.$$

For the integral (1) with unbounded domain, the method leads to an infinite series which should be truncated in practice. However polynomials do not decay at infinity, so Filon method is not applicable in this case.

2.2 Levin collocation method

Another method that is very effective in approximation of highly oscillatory integrals is Levin method, that tries to find $F(x)$ with

$$\int f(x) e^{i\omega g(x)} \, dx = F(x) e^{i\omega g(x)}.$$

Derivation from both sides leads to

$$f(x) = F'(x) + i\omega F(x) g'(x). \quad (2)$$

Eq. (2) is a non-oscillatory first order ODE, which can be approximated with a desired accuracy; in the literature the collocation method is suggested for solving (2). Then

$$I - (-F(0)e^{i\omega g(0)}) = O(\omega^{-s-1}),$$

where the integer $s > 0$ is associates with the multiplicities of the deployed collocation method [5, Theorem 7.1.2]. However, for unbounded integrals (1), Levin’s method results in slower rate of convergence if $f$ vanishes at infinity only polynomially.

2.3 Steepest descent method

The idea behind the steepest descent method is to replace the integration interval with a path in the complex plane, in such a way that the oscillatory factor of integrand decays exponentially along this path. This method is discussed in details for integral (1) with bounded domain $\mathcal{I} := [a, b]$ in [2] and references therein. In [4] the author extended this method to unbounded interval $[0, +\infty)$. In the following, we give a brief report of [4]; only the main results and theorems are given and the proofs are omitted.

Assume that $g$ has no stationary points in the integration interval $[0, \infty)$, i.e, $g'(x) \neq 0, x \geq 0$. Assume also that the equation $g(x) = z$ is easily solvable analytically. Then $h(p) := g^{-1}(g(0) - ip)$ uniquely defines the analytic function $h(p)$ for $p \geq 0$. The oscillatory factor $\exp(i\omega x)$ not only never oscillates but also decays exponentially along the path defined by $h(p)$. This observation leads us to change the route of integration into a path in the complex plane that includes both $h(p)$ and the half-line $[0, \infty)$. 

Theorem 1 Let $f$ and $g$ (with the properties described above) be analytic in a closed simply connected and sufficiently large complex region $D$ which contains the half-line $[0, +\infty)$, and that $g$ is uniquely invertible on $D$. Further assume that the following conditions hold:

(a) $f(z) = f_1(z)f_2(z)$, where $f_1(z)$ is an analytic bounded function in $D$ with $f_1(x) \in L^1([0, \infty))$, and $f_2(z)$ is an analytic function in $D$ vanishing as $|z| \to \infty$.

(b) $|g^{-1}(z)|$ tends to infinity, with the restriction $|g^{-1}(z)| = \mathcal{O}(e^{\omega_0|z|})$ for some $\omega_0 > 0$, as $|z| \to +\infty$.

Then

$$I[f] = e^{i\omega g(0)} \int_0^\infty f(h(p))e^{-\omega p}h'(p) \, dp. \tag{3}$$

The proof can be found in [4].

Integral (3) is nonoscillatory and can be approximated with a desired accuracy using the $n$-point Gauss-laguerre rule; that is

$$e^{i\omega g(0)} \int_0^\infty f(h(p))e^{-\omega p}h'(p) \, dp \approx \frac{e^{i\omega g(0)}}{\omega} \sum_{j=1}^n w_{n,j} f \left( h \left( \frac{x_{n,j}}{\omega} \right) \right) h' \left( \frac{x_{n,j}}{\omega} \right), \tag{4}$$

where $w_{n,j}$ and $x_{n,j}$ are weights and abscissas, respectively, for the Gauss-laguerre quadrature formula.

The proof of the following theorem is exactly the same as that of Theorem 3.6 in [2].

Theorem 2 Approximation (4) has an error that scales as $\mathcal{O}(\omega^{-2n-1})$.

Next we extend the method in the presence of stationary points. Let $f$ and $g$ be analytic in a complex region $D$, as described in Theorem 1. Consider the case when the equation $g'(z) = 0$ has a finite number of zeros in $D$, and they are all in $[0, \infty)$, i.e,

$$\forall z \in \mathbb{C} \setminus \mathbb{R}, \quad f(z) \neq 0, \quad \text{but} \quad \exists \xi_1, \ldots, \xi_l \geq 0 \quad \text{such that} \quad f(\xi_j) = 0, \quad j = 1, \ldots, l.$$  

Set $b := \max_j \xi_j$. Define $r_j := (\min_{k>1} g^{(k)}(\xi_j) \neq 0) - 1$ and $r := \max_j r_j$. Then integral (1) can be divided as

$$\int_0^\infty u(x) \, dx = \int_0^b u(x) \, dx + \int_b^\infty u(x) \, dx.$$  

The integral on $[0, b]$ can be approximated by a numerical steepest descent method described in [2]. According to Theorem 4.6 of [2], the quadrature error is of the order $\mathcal{O}(\omega^{-2n-1/(r+1)})$, where $n$ is the number of the quadrature points. The integral on $[b, \infty)$ equals $\int_0^\infty u(x + b) \, dx$, which can be approximated by (4) with an error of the order $\mathcal{O}(\omega^{-2n-1})$ (see Theorem 2).

Therefore, the total error for approximation of $\int_0^\infty u(x) \, dx$ can be of the order $\mathcal{O}(\omega^{-2n-1/(r+1)})$.

3 Numerical results

Here, with some numerical examples, we illustrate the theoretical discussions of the previous section.

As the first example, consider (1) with $f(x) = 1/(1 + x^2)$ and $g(x) = \exp(\sqrt{x})$. Regarding the assumptions and conditions of Theorem 2, $f$ satisfies (a) as discussed in Remark 2. Moreover, $g^{-1}(x) = (\ln x)^2$, satisfying (b). Hence, we expect an error of the order $\mathcal{O}(\omega^{-2n-1})$ according to Theorem 2. In Table 1, absolute errors of the approximation (4) for some values of $n$ and $\omega$ are
Table 1: Absolute error and the error rate $\log_2(E_{320}/E_{640})$ of the first example with $g(x) = \exp(\sqrt{x})$.

<table>
<thead>
<tr>
<th>$\omega$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
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<td>4.65E-5</td>
<td>1.71E-7</td>
<td>9.54E-10</td>
<td>4.71E-11</td>
<td>1.77E-12</td>
<td>7.60E-14</td>
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<td>80</td>
<td>5.85E-6</td>
<td>5.16E-9</td>
<td>5.01E-12</td>
<td>9.31E-14</td>
<td>7.96E-16</td>
<td>6.28E-18</td>
<td>1.38E-19</td>
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<td>4.97E-12</td>
<td>2.27E-16</td>
<td>3.53E-19</td>
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<td>11.0</td>
<td>13.5</td>
<td>15.0</td>
</tr>
</tbody>
</table>

Table 2: Absolute error and the error rate $\log_2(E_{320}/E_{640})$ of the second example with $g(x) = x^4$.

<table>
<thead>
<tr>
<th>$\omega$</th>
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<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
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<tr>
<td>640</td>
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<td>1.33E-32</td>
<td>1.51E-36</td>
</tr>
<tr>
<td>Rate</td>
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<td>5.0</td>
<td>7.0</td>
<td>9.0</td>
<td>11.0</td>
<td>13.0</td>
<td>15.0</td>
</tr>
</tbody>
</table>

brought. The last row shows the rates of error corresponding to $\omega = 320$ and $\omega = 640$ for various $n$, that is

$$\text{Rate} = \log_2 \left( \frac{E_{320}}{E_{640}} \right)$$

As the second example, we consider $f(x) = 1/(1 + (x - 1)^2)$ and $g(x) = x^4$ on $[1, \infty)$. The discussion on this example is exactly the same as that of the first one. The numerical results are brought in Table 2.

References


