United Arab Emirates University
Faculty of Science
Department of Mathematical Sciences

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# MATH 140: LINEAR ALGEBRA I MIDTERM EXAM FALL 2011

Date and time : 31/10/2011, 7-8pm

Student Name: ..........X..........
Student ID: ..........XXXXXXXXXX..........
Section: ..........08..........

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Grade: ......../20

**Question 1.**[3 marks] For each of the following statements determine whether it is true or false.

(1) An $n \times n$ matrix $A$ is invertible if and only if the reduced echelon form is the identity matrix $I_n$. True ☑ False ☐

(2) For any two matrices $A$ and $B$, we have $ABA^{-1} = B$. True ☐ False ☑

(3) For any $A$, the linear system $AX = O$ has at least one solution. True ☑ False ☐

(4) For any $A$ and $B$, the linear system $AX = B$ has at least one solution. True ☐ False ☑

**Question 2.**[1 mark] Which of the following is (are) NOT TRUE?

(a) $|A + B| = |A| + |B|$

(b) $|AB| = |A||B|$

(c) $tr(A + B) = tr(A) + tr(B)$

(d) $(A + B)^2 = A^2 + 2AB + B^2$

**Question 3.**[1.5 mark] Determine the values of $t$ for which the matrix $A = \begin{pmatrix} 0 & t-1 & 1 \\ 0 & t & t \\ 1 & 0 & t \end{pmatrix}$ is singular?

t= 0 or 2

A is singular $\Leftrightarrow |A| = 0$

$\Leftrightarrow \begin{vmatrix} 0 & t-1 & 1 \\ 0 & t & t \\ 1 & 0 & t \end{vmatrix} = 0$

$\Leftrightarrow \begin{vmatrix} t-1 & 1 \\ t & t \end{vmatrix} = 0 \Leftrightarrow t^2 - 2t = 0$

$\Leftrightarrow t = 0$ or $t = 2$

**Question 4.**[1.5 mark] Let $A = \begin{pmatrix} 1 & x \\ 0 & x^2 \end{pmatrix}$. Determine the values of $x$ for which the matrix $A^2$ is symmetric?

$x =$ ....0..........

$$A^2 = \begin{pmatrix} 1 & x \\ 0 & x^2 \end{pmatrix}\begin{pmatrix} 1 & x \\ 0 & x^2 \end{pmatrix} = \begin{pmatrix} 1 & x+x^3 \\ 0 & x^4 \end{pmatrix}$$

$A^2$ symmetric $\Leftrightarrow x + x^3 = 0 \Leftrightarrow x(1+x^2) = 0$
$\Leftrightarrow x = 0$

**Question 5.**[4 marks] Solve the linear system: $\begin{cases} x_1 - x_2 + x_3 - x_4 = 0 \\ x_1 + 4x_2 - x_3 + 3x_4 = 0 \\ x_1 + x_2 - x_3 = 0 \end{cases}$

we use Gauss- Jordan Method

$$A = \begin{pmatrix} 1 & -1 & 1 & -1 \\ 1 & 4 & -1 & 3 \\ 1 & 1 & -1 & 0 \end{pmatrix} \xrightarrow[R_3 - R_1]{R_2 - R_1} \begin{pmatrix} 1 & -1 & 1 & -1 \\ 0 & 5 & -2 & 4 \\ 0 & 2 & -2 & 1 \end{pmatrix} \xrightarrow{R_2 \times \frac{1}{5}} \begin{pmatrix} 1 & -1 & 1 & -1 \\ 0 & 1 & -\frac{2}{5} & \frac{4}{5} \\ 0 & 2 & -2 & 1 \end{pmatrix}$$

$$\xrightarrow[R_1 + R_2]{R_3 - 2R_2} \begin{pmatrix} 1 & 0 & \frac{3}{5} & -\frac{1}{5} \\ 0 & 1 & -\frac{2}{5} & 4/5 \\ 0 & 0 & -6/5 & -3/5 \end{pmatrix} \xrightarrow{R_3 \times \frac{-5}{6}} \begin{pmatrix} 1 & 0 & \frac{3}{5} & -\frac{1}{5} \\ 0 & 1 & -2/5 & 4/5 \\ 0 & 0 & 1 & \frac{1}{2} \end{pmatrix}$$

$$\xrightarrow[R_1 - \frac{3}{5}R_3]{R_2 + \frac{2}{5}R_3} \begin{pmatrix} 1 & 0 & 0 & -\frac{1}{2} \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & \frac{1}{2} \end{pmatrix} = A_R \text{ : reduced form.}$$

so $x_1, x_2$ and $x_3$ are dependent unknowns
$x_4$ is independent.

set $x_4 = t$ then $x_1 = \frac{1}{2}t$, $x_2 = -t$, $x_3 = -\frac{1}{2}t$

$$\Rightarrow X = t \cdot \begin{pmatrix} \frac{1}{2} \\ -1 \\ -\frac{1}{2} \\ 1 \end{pmatrix}$$

**Question 6.** [5 marks] Consider the matrix, $A = \begin{pmatrix} 1 & 3 & 2 \\ 2 & 8 & 8 \\ -2 & -5 & -3 \end{pmatrix}$

(a) Find $A^{-1}$.

use the formula $A^{-1} = \frac{1}{\det A} \operatorname{adj}(A)$. for example.

$$\det A = \begin{vmatrix} 1 & 3 & 2 \\ 2 & 8 & 8 \\ -2 & -5 & -3 \end{vmatrix} = -2 \neq 0$$

$$C_{11} = \begin{vmatrix} 8 & 8 \\ -5 & -3 \end{vmatrix} = +16$$

$$C_{21} = \begin{vmatrix} 3 & 2 \\ -5 & -3 \end{vmatrix} = -9+10 = 1 \quad , \; \ldots\ldots$$

$$A^{-1} = \frac{1}{-2} \begin{pmatrix} 16 & -1 & 8 \\ -10 & +1 & -4 \\ 6 & -1 & 2 \end{pmatrix} = \begin{pmatrix} -8 & 1/2 & -4 \\ 5 & -1/2 & 2 \\ -3 & 1/2 & -1 \end{pmatrix}$$

(b) Use your result from question (a) to solve the system $\begin{cases} x_1 + 3x_2 + 2x_3 & = 0 \\ 2x_1 + 8x_2 + 8x_3 & = 4 \\ -2x_1 - 5x_2 - 3x_3 & = 2 \end{cases}$

$AX = B$ since $A = \begin{pmatrix} 1 & 3 & 2 \\ 2 & 8 & 8 \\ -2 & -5 & -3 \end{pmatrix}$ is invertible

then the system has a unique solution

$$X = A^{-1} B \iff$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -8 & \frac{1}{2} & -4 \\ 5 & -1/2 & 2 \\ -3 & 1/2 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 4 \\ 2 \end{pmatrix} = \begin{pmatrix} -6 \\ 2 \\ 0 \end{pmatrix}$$

**Question 7.**[4 marks]

Use Cramer's rule to solve the following system of linear equations $\begin{cases} x_1 - x_2 + x_3 &= 1 \\ x_1 + x_2 - 2x_3 &= 0 \\ x_1 + x_2 + x_3 &= 0 \end{cases}$

$$A = \begin{pmatrix} 1 & -1 & 1 \\ 1 & 1 & -2 \\ 1 & 1 & 1 \end{pmatrix}$$

$$|A| = \begin{vmatrix} 1 & -1 & 1 \\ 1 & 1 & -2 \\ 1 & 1 & 1 \end{vmatrix} = 6.$$

By Cramer's rule

$$x_1 = \frac{\begin{vmatrix} 1 & -1 & 1 \\ 0 & 1 & -2 \\ 0 & 1 & 1 \end{vmatrix}}{6} = \frac{\begin{vmatrix} 1 & -2 \\ 1 & 1 \end{vmatrix}}{6} = \frac{1}{2}$$

$$x_2 = \frac{\begin{vmatrix} 1 & 1 & 1 \\ 1 & 0 & -2 \\ 1 & 0 & 1 \end{vmatrix}}{6} = \frac{-\begin{vmatrix} 1 & -2 \\ 1 & 1 \end{vmatrix}}{6} = -\frac{1}{2}$$

$$x_3 = \frac{\begin{vmatrix} 1 & -1 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \end{vmatrix}}{6} = 0$$

$$X = \begin{pmatrix} 1/2 \\ -1/2 \\ 0 \end{pmatrix}.$$