UAEU College of Science كلية العلوم

Department of Mathematical Sciences

جامعة الإمارات العربية المتحدة
United Arab Emirates University

قسم علــوم الرياضيـــــات

MATH 2220: Linear Algebra for Eng.,

Final Exam Fall 2022

Date: 13/12/2022, Time: 8:30-10:30AM

Student Name: Sol. Key

Student ID: ____________________ **Attendance Number:** _____

Question	Part 1	Part II			Grade
		1-2-3	4	5	
CLO	1	2	3	4	
Marks	....../18	....../12	....../4	....../6	.... / 40

- Show all the steps of your solution for each question in Part II.
- Use only Blue or Black pen, neither pencil nor colored.
- Graphics and Programming Calculators are not allowed.

Section	Instructor Name	Section	Instructor Name
01	Prof. N. Chbili	54	Prof. N. Chbili
03	Prof. A. Al Rawashdeh	55	Dr. J. Gong
51	Dr. Z. Balogh	56	Prof. V. Bodi
52	Prof. F. Mukhamedov	57	Prof. A. Al Rawashdeh
53	Prof. A. Zubkov	58	Dr. U. Goginava

Part 1. [18 marks] (2 marks each)

For each of the questions below, put a circle around the correct answer.

1. Given $A = \begin{bmatrix} 3 & 2 & 2 \\ 1 & 2 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix}$, then $tr(AB^T - I_2) =$

(A) 2 **(B)** 4 **(C)**6 **(D)** 8 **(E)** None of the above

2. Given $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 2$, then $\begin{vmatrix} 2a & 2b & 2c \\ g & h & i \\ d+3g & e+3h & f+3i \end{vmatrix} =$

(A) -4 **(B)** -2 **(C)** 2 **(D)** 4 **(E)** None of the above

3. Which of the following is a subspace of $\mathbb{R}^3$?

(A) $\{(x, y, 1); x$ and y are any real numbers$\}$

(B) $\{(x, y, 2x); x$ and y are any real numbers$\}$

(C) $\{(x, x^2, 0); x$ is any real number$\}$

(D) $\{(x, y, xy); x$ and y are any real numbers$\}$

(E) $\{(x, x, x+1); x$ is any real number$\}$

4. Determine for which values of k, the 3 polynomials $q_1(x) = 2$, $q_2(x) = x + x^2$ and $q_3(x) = 1 + x + kx^2$ are linearly independent.

(A) $k = 1$ and $k = -1$

(B) $k = 0$ and $k = 1$

(C) $k \neq 0$

(D) $k \neq 1$

(E) k is any real number

5. Let A be an $(n \times n)$-matrix whose nullity is 0. Which of the following is NOT correct?

(A) A is invertible.

(B) The row vectors of A are linearly independent.

(C) The reduced row echelon form of A is I_n.

(D) $\text{rank}(A) = n$.

(E) 0 is an eigenvalue of A.

6. Let W be the set of 2×2 symmetric matrices whose trace is zero, then

(A) W is a subspace of $M_{2,2}$ and $\dim(W) = 1$.

(B) W is a subspace of $M_{2,2}$ and $\dim(W) = 2$.

(C) W is a subspace of $M_{2,2}$ and $\dim(W) = 3$

(D) W is a subspace of $M_{2,2}$ and $\dim(W) = 4$.

(E) W is not a subspace of $M_{2,2}$.

7. Given $A = \begin{bmatrix} a & 2 & 2 \\ 1 & 2 & 0 \\ 1 & 1 & 1 \end{bmatrix}$. For which values of a is $v = (2, 1, 1)$ an eigenvector of A?

(A) 0 **(B)** 1 **(C)** 2 **(D)** 3 **(E)** None of the above

8. The characteristic equation of the matrix $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ is

(A) $\lambda(\lambda - 1)(\lambda - 2) = 0$.

(B) $(\lambda - 1)^2(\lambda - 2) = 0$

(C) $(\lambda - 1)(\lambda - 2)^2 = 0$.

(D) $\lambda(\lambda - 2)^2 = 0$.

9. Given u and v two vectors in an inner product space, such that $||u|| = 2$, $||v|| = 2$ and $< u, v >= 1$. Then $< u + 2v, 2u - v >=$

(A) 0 **(B)** 1 **(C)** 2 **(D)** 3 **(E)** None of the above

Part II. [22 marks]

Show all the steps of your solution for each of the following questions.

Question 1. **[4 marks]** Use the Gauss-Jordan Elimination Method to solve the following system of linear equations:

$$\begin{cases} x_1 + x_2 + x_3 + x_4 &= 1 \\ 2x_1 + 2x_2 + x_3 &= 1 \\ x_1 - x_2 + x_3 - x_4 &= 0 \end{cases}$$

Augmented matrix $\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 1 & 0 & 1 \\ 1 & -1 & 1 & -1 & 0 \end{array}\right]$ 0,5

Find RREF of A

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 1 & 0 & 1 \\ 1 & -1 & 1 & -1 & 0 \end{array}\right] \xrightarrow[R_3 - R_1]{R_2 - 2R_1} \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & -1 & -2 & -1 \\ 0 & -2 & 0 & -2 & -1 \end{array}\right] \xrightarrow{R_3 \leftrightarrow R_2} \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 0 & -2 & 0 & -2 & -1 \\ 0 & 0 & -1 & -2 & -1 \end{array}\right]$$

$$\xrightarrow{R_2 \times -\frac{1}{2}} \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1/2 \\ 0 & 0 & -1 & -2 & -1 \end{array}\right] \xrightarrow{R_1 - R_2} \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & \frac{1}{2} \\ 0 & 1 & 0 & 1 & 1/2 \\ 0 & 0 & -1 & -2 & -1 \end{array}\right] \xrightarrow{R_3 \times -1} \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & \frac{1}{2} \\ 0 & 1 & 0 & 1 & 1/2 \\ 0 & 0 & 1 & 2 & 1 \end{array}\right]$$

$$\xrightarrow{R_1 - R_3} \left[\begin{array}{cccc|c} 1 & 0 & 0 & -2 & -\frac{1}{2} \\ 0 & 1 & 0 & 1 & 1/2 \\ 0 & 0 & 1 & 2 & 1 \end{array}\right]$$ 2

$x_4 = t$ free unknown 0,5

$x_1 = -\frac{1}{2} + 2t$; $x_2 = \frac{1}{2} - t$, $x_3 = 1 - 2t$ 1

general solution $\begin{bmatrix} -\frac{1}{2} + 2t \\ \frac{1}{2} - t \\ 1 - 2t \\ t \end{bmatrix}$; t any real number.

Question 2. [3 marks] Let W be the subspace of $\mathbb{R}^4$ spanned by $v_1 = (1, 2, 0, 1)$, $v_2 = (2, 4, 3, 4)$, $v_3 = (1, 2, 3, 4)$ and $v_4 = (0, 0, 2, 2)$.

(a) Find a basis for W.

consider the matrix $A = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 2 & 4 & 3 & 4 \\ 1 & 2 & 3 & 4 \\ 0 & 0 & 2 & 2 \end{bmatrix}$; $W = \text{row}(A)$

Let us find a basis for row(A) 0,5

$$\begin{bmatrix} 1 & 2 & 0 & 1 \\ 2 & 4 & 3 & 4 \\ 1 & 2 & 3 & 4 \\ 0 & 0 & 2 & 2 \end{bmatrix} \xrightarrow[r_3 - r_1]{r_2 - 2r_1} \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 2 & 2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 2/3 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 2 & 2 \end{bmatrix}$$

$$\xrightarrow[r_4 - 2r_2]{r_3 - 3r_2} \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 2/3 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 2/3 \end{bmatrix} \xrightarrow{r_4 - \frac{2}{3}r_3} \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 2/3 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$ R.E.F 1

A basis for W is
$w_1 = (1, 2, 0, 1)$
$w_2 = (0, 0, 1, \frac{2}{3})$ 0,5
$w_3 = (0, 0, 0, 1)$

(b) Determine the dimension of W.

1

$\dim W = 3$

Question 3. [5 marks] Let $v_1 = (1, 1, 1)$, $v_2 = (1, 1, -1)$ and $v_3 = (1, -1, 2)$ be 3 vectors of $\mathbb{R}^3$.

(a) Show that $S = \{v_1, v_2, v_3\}$ is a basis for $\mathbb{R}^3$.

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & 2 \end{vmatrix} = 1 . \begin{vmatrix} 1 & -1 \\ -1 & 2 \end{vmatrix} - 1 . \begin{vmatrix} 1 & -1 \\ 1 & 2 \end{vmatrix} + 1 . \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix}$$

$$= 1 - 3 + (-2) = -4 \neq 0$$ ①

Since the determinant is non zero then

the vectors v_1, v_2, v_3 form a basis for $\mathbb{R}^3$. ①

(b) Find the coordinates of the vector $w = (0, 1, 0)$ in the basis $S = \{v_1, v_2, v_3\}$.

Solve $c_1 v_1 + c_2 v_2 + c_3 v_3 = w$

$$\Leftrightarrow \begin{cases} c_1 + c_2 + c_3 = 0 \\ c_1 + c_2 - c_3 = 1 \\ c_1 - c_2 + 2c_3 = 0 \end{cases}$$ ①

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & 2 \end{vmatrix} = -4 \neq 0$$ By Cramer's Rule

$$c_1 = \frac{\begin{vmatrix} 0 & 1 & 1 \\ 1 & 1 & -1 \\ 0 & 1 & 2 \end{vmatrix}}{-4} = \frac{3}{4}$$

$$c_2 = \frac{\begin{vmatrix} 1 & 0 & 1 \\ 1 & 1 & -1 \\ 1 & 0 & 2 \end{vmatrix}}{-4} = \cancel{\frac{4}{4}} -\frac{1}{4} ; \quad c_3 = \frac{\begin{vmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & -1 & 0 \end{vmatrix}}{-4} = \frac{-1}{2}$$

Then $(w)_S = \left(\frac{3}{4}, -\frac{1}{4}, -\frac{1}{2}\right)$. ②

Question 4. [4 marks] Let $A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$.

Given that the eigenvalues of A are $0, 1$ and 2, find an invertible matrix P that diagonalizes A.

Eigen space E_0: solve HLS $Ax = 0$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$x_2 = t$ is a free unknown

$x_1 = -t$, $x_3 = 0$

general solution $\begin{bmatrix} -t \\ t \\ 0 \end{bmatrix} = t \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ Let $v_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$

Basis for eigenspace E_0 is $v_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ 1

Eigenspace E_1: Solve HLS $(I_3 - A)x = 0$

$$\left[\begin{array}{ccc|c} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & -1 & 0 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$x_3 = t$ free unknown

$x_1 = 0$, $x_2 = 0$

general solution $\begin{bmatrix} 0 \\ 0 \\ t \end{bmatrix} = t \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, Let $v_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

Basis for E_1 is $v_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ 1

Eigenspace E_2 Solve HLS $(2I_3 - A)x = 0$

we find $v_3 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$ is a basis for the Eigenspace E_2. 1

The matrix A is diagonalizable and $P = \begin{bmatrix} -1 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix}$ 1

$P^{-1}AP = D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

Question 5. [6 marks] For $p = a_0 + a_1x + a_2x^2$ and $q = b_0 + b_1x + b_2x^2$, the standard inner product in the vector space P_2 is defined by:

$$< p, q >= a_0b_0 + a_1b_1 + a_2b_2.$$

(a) Given $p = 1 + 2x + x^2$ and $q = 2 - x + 2x^2$. Evaluate the following:

(1) $< p, q > = 1\times 2 + 2\times -1 + 1\times 2 = 2$

1

(2) $||p|| = \sqrt{<p,p>} = \sqrt{1^2+2^2+1^2} = \sqrt{6}$

1

(3) $||q|| = \sqrt{<q,q>} = \sqrt{2^2+(-1)^2+2^2} = \sqrt{9} = 3$

1

(4) $\cos\theta$, where θ is the angle between p and q.

$$\cos\theta = \frac{<p,q>}{\|p\|\,\|q\|} = \frac{2}{3\sqrt{6}} \approx 0.2721$$

1

(5) $d(p, q) = \|p-q\| = \|-1+3x-x^2\| = \sqrt{(-1)^2+3^2+(-1)^2}$

$= \sqrt{11}$

1

(b) Find the value of the real number t for which the polynomials p and $p + tq$ are orthogonal.

p and $p+tq$ are orthogonal $\Leftrightarrow$ $<p, p+tq> = 0$

1

$<p, p+tq> = \|p\|^2 + t<p,q>$

$= 6 + t\times 2$

$6+2t = 0 \Leftrightarrow t = -3.$