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Nonparametric weighted symmetry tests

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Abstract: Weighted symmetry is an extension of the classical notion of symmetry in which the tails of a distribution are similar, up to a scaling factor. The authors develop test statistics of weighted symmetry based on empirical processes. The finite-dimensional distributions of the proposed statistics are either nonparametric or conditionally nonparametric, according as the parameters of weighted symmetry are known or estimated. Asymptotically, the distributions of the processes behave like Brownian bridges or motions, leading to familiar distributions for the proposed test statistics. The authors also establish the asymptotic normality of Hodges–Lehmann type estimators based on a generalization of the Wilcoxon signed rank test. Furthermore, they propose density estimators in that setting.

Tests de symétrie pondérée non paramétriques

Résumé : La symétrie pondérée est une généralisation de la notion classique de symétrie dans laquelle les queues d'une distribution sont similaires, à un facteur près. Les auteurs déduisent de processus empiriques des statistiques permettant de tester la symétrie pondérée. À taille finie, les lois des statistiques étudiées sont soit non paramétriques, soit conditionnellement non paramétriques, selon que les paramètres régissant la symétrie pondérée sont connus ou estimés. Asymptotiquement, les processus empiriques se comportent comme le mouvement ou le pont brownien, ce qui conduit à des lois limites familières pour les statistiques proposées. Les auteurs établissent aussi la normalité asymptotique d'estimateurs de type Hodges–Lehmann obtenus par généralisation de la statistique des rangs signés de Wilcoxon. Ils proposent de plus des estimateurs de la densité adaptés à ce contexte.

1. INTRODUCTION

Testing whether or not a set of data is drawn from a symmetric distribution is one of the basic problems of nonparametric statistics. In practice, one often has to perform such a test in order to decide which further analysis should be carried out. For instance, in the well-known problem of paired comparisons, a lack of treatment effect reduces to a symmetry test. Robust methods heavily rely on the assumption of symmetry and perform quite poorly under asymmetric models. Also, the choice of a natural measure of location is far from being a simple task when the underlying distribution is asymmetric. This is why symmetry tests have been the subject of many papers; see, e.g., Eubank, LaRiccia & Rosenstein (1992) and the references therein.

Symmetry can rarely be assumed in the huge and growing area of statistical application to financial and economic data. In particular, it is often argued that prices respond less to "positive news" than to "negative news;" see, e.g., Arden, Holly & Turner (1997) and Shirvani & Wilbratte (1999). It may also be shown that inflation is more responsive to positive monetary shocks than to negative shocks; see Rhee (1995). The same phenomenon holds for returns and other economic measures. In the econometric literature, this is usually referred to as price stickiness, asymmetric responses or the ratchet effect. It suggests that relaxing the symmetry hypothesis is crucial for these applications. Lack of symmetry in put and call option prices is also indicative of extremal events, as discussed in Bates (1991).

Hereafter, the notion of symmetry is relaxed in order to accommodate data sets that may exhibit a certain degree of asymmetry. Evidence will be given to show that the extended class provides, in particular, a good fit for economic and financial data. As a first illustration, consider the weekly differences in (unleaded) gasoline prices (in cents per liter) in Québec City (Canada).

Figure 1 depicts the histogram and density kernel estimate of the 109 nonzero values of this variable in the period extending from 27 December 1999 to 25 March 2002. Since zero is a natural cutoff between increases and decreases, the economics literature (see Brown & Yücel (2000) and the references therein) attaches an importance to this point and to the symmetry of positive and negative price reactions. The graph confirms consumers' general perception that the asymmetry in price variation is not in their favour, i.e., that price hikes are usually larger than price reductions. This is apparent in Figure 1 and is also demonstrated by some classical symmetry tests provided in Table 1.

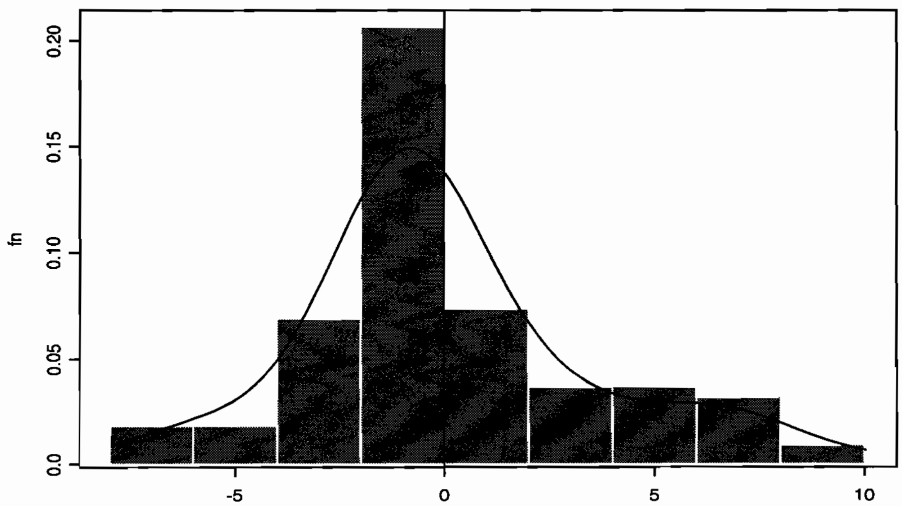

FIGURE 1: Histogram of weekly variations of gasoline prices in the city of Québec (Canada).

TABLE 1: Some classical symmetry tests for the gas data. The Kolmogorov–Smirnov statistic is the supremum of the empirical rank symmetry process proposed by Shorack & Wellner (1986, p. 743).

Symmetry test	value	P-value
Sign test	41	0.0128
Wilcoxon signed rank test	2628	0.2601
Kolmogorov–Smirnov test	3.4482	0.001

Symmetry about zero is rejected by two out of the three tests. Moreover, as Figure 1 suggests, these observations seem to be symmetric with respect to a point smaller than zero. One can argue that another possible explanation is that this is just an optical illusion due to the discontinuity at zero of the underlying density. This example will be revisited in Section 4, where it will be shown that these data fit nicely the extended notion of symmetry defined next. Moreover, simulations of random variables with discontinuous split normal densities show that the histograms obtained look quite similar to the above figure.

DEFINITION 1 (weighted symmetry). *A random variable X with distribution function F is said to be weighted symmetric about θ if and only if there exist two constants $p \in (0, 1)$ and $\omega > 0$ such that for all $x > 0$,*

$$pF(\theta - x^-) = (1-p)\{1 - F(\theta + \omega x)\}, \tag{1}$$

where $F(y^-)$ stands for the left limit of F at y.

When needed, the terminology (p, ω)-symmetry will be used to refer to weighted symmetry as defined by (1). Note that the choices $p = 1/2$ and $\omega = 1$ yield classical symmetry; thus the latter is more restrictive than weighted symmetry.

One of the most important properties of weighted symmetry is that it is equivalent to the split, or two-piece models that can be constructed by joining two halves of the same distribution with different weights assigned to the left and right parts. For applications of split models, see Stigler & Kindhal (1970), Gibbons, Johnson & Mylroie (1975), Aigner, Amemiya & Poirier (1976), Lefrançois (1989) or Scallan (1995). Weighted symmetry has recently been used in option pricing by Gouriéroux & Monfort (2002), where an explicit pricing formula is given for the "skewed Laplace distribution."

An easy way to generate two-piece models is as follows. Let Z be a nonnegative random variable, and consider the random variable X given by $X = \theta - Z$ with probability $1 - p$, and $X = \theta + \omega Z$ with probability p, where $\theta \in \mathbb{R}$, $p \in (0, 1)$, and $\omega > 0$ are given constants.

This representation suggests that X models phenomena that react differently to a positive or negative shock Z. In particular, the reaction to a positive shock of amplitude Z is ωZ, while the reaction to a negative shock of amplitude Z is $-Z$. Note that the reaction to negative shocks is normalized to unity to avoid overparameterizations. In this sense, saying that X is weighted symmetric amounts to saying that there is some hidden factor Z to which the variable X reacts differently depending on the sign of this factor, which is the same behaviour noted earlier for financial and economic data and known as the ratchet effect. This suggests that these distributions can be used to model the asymmetric economical phenomena discussed above. Note also that $\mathrm{E}(X) = \theta + (p\omega + p - 1)\mathrm{E}(Z)$, showing that $\mathrm{E}(X) > \theta$ if and only if $p > 1/(1+\omega)$, whenever $\mathrm{P}(Z = 0) < 1$. Empirical evidence for these findings shall be given in Section 4. To impart the flavour of the asymmetry allowed by these models, some two-piece Gaussian densities are shown in Figure 2.

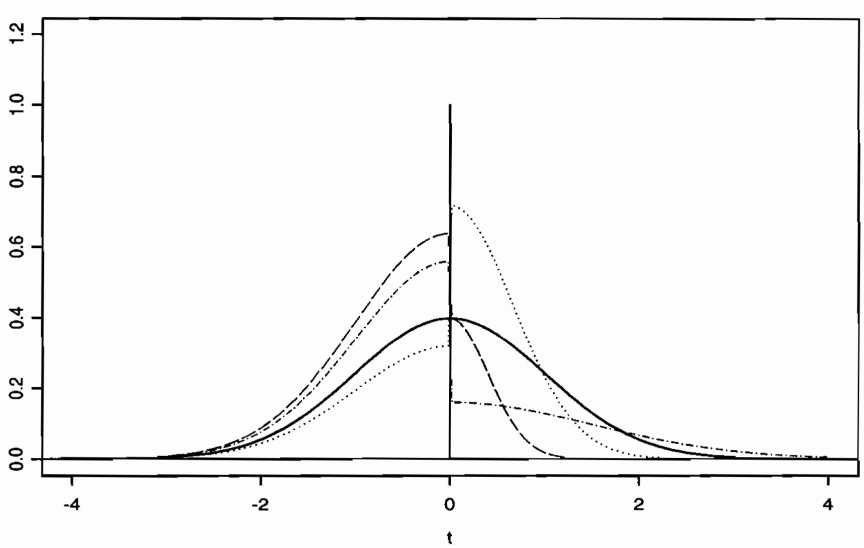

FIGURE 2: Some Gaussian split models.

If a weighted symmetric distribution F has a density f, the latter will not be continuous at its weighted symmetry point θ in general. To see this, denote by $f(\theta^-)$ and $f(\theta^+)$ the left and right derivatives of F at θ and use (1) to get

$$pf(\theta^-) = \omega(1-p)f(\theta^+).$$

Thus f is continuous at θ whenever it has a dip at θ, in which case $f(\theta^-) = f(\theta^+) = 0$, or when the parameters p and ω fulfill $p = (1-p)\omega$. This condition is satisfied by classical symmetric models for which $p = 1/2$ and $\omega = 1$.

In classical symmetry, many statistical procedures take advantage of the fact that a random variable X with continuous distribution is symmetric with respect to a point θ if and only if $\mathrm{P}\{\mathrm{sign}(X-\theta) = 1\} = 1/2$ and $\mathrm{sign}(X-\theta)$ and $|X-\theta|$ are two independent random variables. A similar characterization holds for weighted symmetry. Indeed, Abdous & Rémillard (1995) showed that if F is continuous at θ, then F is (p,ω)−symmetric about θ if and only if $\mathrm{P}\{\mathrm{sign}(X-\theta) = 1\} = p$ and the random variables $|X-\theta|_\omega$ and $\mathrm{sign}(X-\theta)$ are independent, where

$$|X-\theta|_\omega = |X-\theta|\big\{\mathbb{I}(X>\theta) + \omega\mathbb{I}(X \le \theta)\big\},$$

and $\mathrm{sign}(X-\theta) = \mathbb{I}(X>\theta) - \mathbb{I}(X<\theta)$.

An obvious extension of this characterization is given by the following proposition, which is essential in the sequel.

PROPOSITION 1. *A random variable X is (p,ω)-symmetric about θ if and only if*

$$\mathrm{P}\big\{\mathrm{sign}(X-\theta) = 1 \,|\, X-\theta \neq 0\big\} = p,$$

and the random variables $|X-\theta|_\omega$ and $\mathrm{sign}(X-\theta)$ are conditionally independent given $X-\theta \neq 0$.

Remark 1. In practice, if θ is known and $0 < \mathrm{P}(X_i = \theta) < 1$, then one should discard observations that are equal to θ. Therefore, there is no loss of generality if one assumes that the law of X is continuous at θ.

In the sequel, we will use the above characterization to develop statistical procedures enabling us to verify whether a distribution function F is weighted symmetric or not. In fact, generalizing Wilcoxon's signed-rank statistic, Abdous & Rémillard (1995) proposed the following statistic to test weighted symmetry:

$$AR_n = \frac{1}{n^2}\sum_{i=1}^{n} \mathrm{sign}(X_i-\theta)\,\mathrm{rank}(|X_i-\theta|_\omega), \tag{2}$$

where $X_1,\ldots,X_n$ is a random sample drawn from F.

The next section discusses the identifiability question and provides nonparametric estimates of the parameters. First, using the Hodges–Lehmann technique with the statistic AR_n, we shall give an estimate of the symmetry point θ for the case where p and ω are known. Next, we will discuss an estimate of p and ω for the case where θ is known. The section will conclude with a discussion of the simultaneous estimation of all weighted symmetry parameters.

In Section 3, we shall define an empirical distribution function whose mean is related to AR_n. Then we will introduce an empirical process that will be used to define powerful statistics for testing weighted symmetry. An important feature is that if p is known, then the finite-dimensional distributions of the proposed test statistics are nonparametric and can be calculated for any sample size. When p is unknown and is estimated by a sample proportion $\hat{p}_n$, then conditional on that value, the finite-dimensional distributions of the proposed test statistics are also nonparametric. Limiting distributions can be specified in both cases. The gas data example will be continued in Section 4, where weighted symmetric density estimates will be proposed. A second application to stock market indices will also be discussed. The mathematical derivations are postponed to Section 5.

2. HODGES–LEHMANN ESTIMATION OF WEIGHTED SYMMETRY PARAMETERS

This section discusses the estimation of weighted symmetry parameters. First, a result outlining the identifiability of the weighted symmetric distribution is presented. Then, the Wilcoxon signed rank statistic AR_n is used to derive Hodges–Lehmann estimates of the parameters ω and θ. Simultaneous estimation of all weighted symmetry parameters is discussed at the end of the section.

2.1. Identifiability of the weighted symmetric distribution.

The identifiability conditions are summarized in the next proposition.

PROPOSITION 2. *Assume that F is weighted symmetric, continuous and admits a density with finitely many discontinuities and finitely many nonzero local maxima or minima in its support. Then F is identifiable.*

Remark 2. Note that the uniform distribution constitutes a pathological case here. In fact, the uniform density has infinitely many local maxima or minima in its support and it can be seen that the uniform distribution on $(0,1)$ is (p,ω)-symmetric about any point θ in its support, provided that $p = 1-\theta$ and $\omega = 1/\theta - 1$.

2.2. Estimation of θ when p and ω are known.

We begin with the location parameter θ, even though this situation should not occur often in applications. Assume that $p \in (0,1)$ and $\omega > 0$ are given constants. Suppose that the underlying probability distribution is (p,ω)-symmetric with respect to an unknown value θ_0. For arbitrary θ, set

$$AR_n(\theta) = \frac{1}{n^2}\sum_{i=1}^{n} \operatorname{sign}(X_i - \theta)\,\operatorname{rank}(|X_i - \theta|_\omega).$$

Since $AR_n(\,\cdot\,)$ is nonincreasing

$$\lim_{\theta\to-\infty} AR_n(\theta) = \frac{n+1}{2n} \quad\text{and}\quad \lim_{\theta\to\infty} AR_n(\theta) = -\frac{n+1}{2n},$$

the Hodges–Lehmann estimator of θ_0 is well defined and given by

$$\hat{\theta}_n = \frac{\theta_{n,1} + \theta_{n,2}}{2},$$

where

$$\theta_{n,1} = \sup\left\{\theta : AR_n(\theta) > \frac{n+1}{2n}(2p-1)\right\}, \quad \theta_{n,2} = \inf\left\{\theta : AR_n(\theta) < \frac{n+1}{2n}(2p-1)\right\}.$$

The consistency of this estimator is established in the next theorem.

THEOREM 3. *If F is (p,ω)-symmetric about θ_0 and admits a square integrable density f, then $\sqrt{n}\,(\hat{\theta}_n - \theta_0)$ converges in law to a centered Gaussian random variable with variance $4p(1-p)/(3c_1^2)$, where $c_1 = 2(1+\omega)(1-p)p^{-1}\int_{\theta_0}^{\infty} f^2(z)\,dz$. In particular, the Hodges–Lehmann estimator $\hat{\theta}_n$ is consistent.*

2.3. Estimation of ω when θ is known.

Assume that the parameter ω is unknown, while θ is a given constant. This should be the case in most applications. For any arbitrary $\omega > 0$, denote by $AR_n(\omega)$ the associated Wilcoxon signed rank statistic. As ω varies, $AR_n(\omega)$ is nonincreasing and satisfies

$$\begin{aligned}\lim_{\omega\to 0} AR_n(\omega) &= \hat{p}_n \frac{n+1}{2n} + \frac{\hat{p}_n(1-\hat{p}_n)}{2},\\ \lim_{\omega\to\infty} AR_n(\omega) &= -(1-\hat{p}_n)\frac{n+1}{2n} - \frac{\hat{p}_n(1-\hat{p}_n)}{2},\end{aligned}$$

where $\hat{p}_n$ stands for the proportion of observations $X_i > \theta$. If the underlying distribution is (p, ω_0)-symmetric with respect to a given θ, then the Hodges–Lehmann estimator $\widehat{\omega}_n$ of ω_0 is well defined and is given by $\widehat{\omega}_n = (\omega_{n,1} + \omega_{n,2})/2$, where

$$\omega_{n,1} = \sup\left\{\omega : AR_n(\omega) > \frac{n+1}{2n}(2q_n - 1)\right\},\ \omega_{n,2} = \inf\left\{\omega : AR_n(\omega) < \frac{n+1}{2n}(2q_n - 1)\right\},$$

with $q_n = \hat{p}_n$ if p is unknown, and $q_n = p$ otherwise. The following theorem establishes the asymptotic properties of $\widehat{\omega}_n$.

THEOREM 4. *Suppose that F is a (p, ω_0)-symmetric distribution whose density f is such that*

$$\int |x| f^2(x)\, dx < \infty.$$

Then $\sqrt{n}\,(\widehat{\omega}_n - \omega_0)$ converges in law to a centered Gaussian random variable with variance $\sigma^2 p(1-p)/c_2^2$, where $\sigma^2 = 4/3$ or $1/3$ according as p is known or not, and where

$$c_2 = \frac{2}{\omega_0}\frac{1-p}{p}\int_\theta^\infty (z-\theta) f^2(z)\, dz.$$

Moreover, if p is unknown, then $\sqrt{n}\,(\widehat{\omega}_n - \omega_0)$ and $\sqrt{n}\,(\hat{p}_n - p)$ converge jointly to independent Gaussian variables.

Remark 3. As a consequence, under $\mathcal{H}_0 : p = \omega_0/(1+\omega_0)$, the statistic $\sqrt{n}\,\{\hat{p}_n - \widehat{\omega}_n/(1+\widehat{\omega}_n)\}$ converges in law to a centered Gaussian random variable with variance

$$p(1-p) + \frac{p(1-p)}{3(1+\omega_0)^4 c_2^2}.$$

The latter result can be used to develop a test of continuity of the density at θ under the model of weighted symmetry. Recall that if the density f is continuous at θ and $f(\theta) > 0$, it follows that $\omega/(1+\omega) = p$.

2.4. Simultaneous estimation of weighted symmetry parameters.

Now assume that all three parameters are unknown and that F satisfies the conditions of Proposition 2. An idea presented in Shorack & Wellner (1986, p. 759) for the classical symmetry can then be adapted to our context to yield the following consistent estimate of (θ, ω, p):

$$(\hat{\theta}_n, \widehat{\omega}_n, \hat{p}_n) = \arg\min_{\theta,\omega,p} T_n(\theta, \omega, p),$$

where

$$\begin{aligned} T_n(\theta,\omega,p) &= \int_0^\infty n\big[pF_n(\theta - x^-) - (1-p)\{1 - F_n(\theta + \omega x)\}\big]^2 dx\\ &= \frac{1}{n}\sum_{i=1}^n\sum_{j=1}^n \{p\,\mathbb{I}(X_i \le \theta) - (1-p)\mathbb{I}(X_i > \theta)\}\\ &\quad \times \{p\,\mathbb{I}(X_j \le \theta) - (1-p)\mathbb{I}(X_j > \theta)\} \min\left(\frac{|X_i - \theta|_\omega}{\omega}, \frac{|X_j - \theta|_\omega}{\omega}\right).\end{aligned}$$

The asymptotics of such an estimate are outlined in the next theorem.

THEOREM 5. *If F is identifiable, (p_0, ω_0)-symmetric about θ_0 and admits a density f such that*

$$\int f^2(x)\,dx < \infty, \quad \int |x| f(x)\,dx < \infty, \quad \int |x| f^2(x)\,dx < \infty, \quad \text{and} \quad \int |x|^2 f^2(x)\,dx < \infty,$$

then $\left(\sqrt{n}\,(\hat{\theta}_n - \theta_0), \sqrt{n}\,(\widehat{\omega}_n - \omega_0), \sqrt{n}\,(\hat{p}_n - p_0)\right)$ converges to a multivariate Gaussian random vector with mean zero and covariance matrix $A^{-1}\Sigma A^{-1}$, where the entries of A are given by

$$A_{ij} = \int_0^\infty h_i(x, \theta_0, \omega_0, p_0) h_j(x, \theta_0, \omega_0, p_0)\,dx, \quad 1 \le i, j \le 3,$$

and the entries of Σ are given by

$$\Sigma_{ij} = \int_0^\infty \int_0^\infty p_0 \min\{F(\theta_0 - x), F(\theta_0 - y)\} h_i(x, \theta_0, \omega_0, p_0) h_j(y, \theta_0, \omega_0, p_0)\,dx\,dy,$$

$1 \le i, j \le 3$, and

$$\begin{aligned} h_1(x, \theta_0, \omega_0, p_0) &= p_0(1 + 1/\omega_0) f(\theta_0 - x), \\ h_2(x, \theta_0, \omega_0, p_0) &= x p_0 f(\theta_0 - x)/\omega_0, \\ h_3(x, \theta_0, \omega_0, p_0) &= F(\theta_0 - x)/(1 - p_0). \end{aligned}$$

3. WEIGHTED SYMMETRY PROCESS

Let $X_1, \ldots, X_n$ be a random sample from a distribution F, and suppose that one would like to test whether F is (p, ω)-symmetric about θ or not. To do this, one could use the statistic AR_n defined by (2). One can also try a potentially more powerful approach using empirical processes based on pseudo-observations, as defined in Ghoudi & Rémillard (1998, 2003). In the present case, the mean of these processes is related to the statistic AR_n. In what follows, we will consider two cases: p known and p estimated.

3.1. *Parameters p, θ and ω known.*

The inference will be based on functionals of the weighted symmetry process given by

$$\mathbb{K}_n(t, p) = \frac{\sqrt{n}\left[K_n(t) - \mathrm{E}\{K_n(t)\}\right]}{\sqrt{p(1-p)}}, \quad t \in [-1, 1],$$

where

$$K_n(t) = \frac{1}{n} \sum_{i=1}^n \mathbb{I}(e_{n,i} \le t),$$

and where the pseudo-observations $e_{n,i}$ are defined by

$$e_{n,i} = \frac{1}{n} \operatorname{sign}(X_i - \theta) \operatorname{rank}(|X_i - \theta|_\omega), \quad 1 \le i \le n.$$

If $A_1, \ldots, A_n$ denote the corresponding anti-ranks, i.e., $A_i = k$ if and only if $\operatorname{rank}(|X_k - \theta|_\omega) = i$, then one can write

$$K_n(t) = \frac{1}{n} \sum_{i=1}^n \mathbb{I}(i \varepsilon_{n+1-i} \le nt),$$

where $\varepsilon_{n+1-i} = \operatorname{sign}(X_{A_i} - \theta)$. Setting

$$\delta_i = \mathbb{I}(\varepsilon_i = -1) = \mathbb{I}(X_{A_{n+1-i}} < \theta), \tag{3}$$

one can readily check that

$$\mathbb{K}_n(t,p) = \begin{cases} W_n(1+1/n+[nt]/n), & -1 \le t < 0, \\ W_n(1-[nt]/n), & 0 \le t \le 1, \end{cases} \tag{4}$$

where $[x]$ denotes the integer part of x,

$$W_n(u) = \frac{1}{\sqrt{np(1-p)}} \sum_{i=1}^{[nu]} \{\delta_i - (1-p)\} = \frac{S_{[nu]}}{\sqrt{np(1-p)}}, \tag{5}$$

where $S_k = \sum_{i=1}^{k} \{\delta_i - (1-p)\}$ is the partial sum and $S_0 = 0$.

The following theorem establishes the asymptotic behaviour of $\mathbb{K}_n(t,p)$ when the underlying distribution F is weighted symmetric with known parameters p, ω and θ.

THEOREM 6. *Let $p \in (0,1)$, $\omega > 0$ and $\theta \in \mathbb{R}$ be given constants, and assume that the distribution function F is (p,ω)-symmetric about θ. Then $\delta_1, \ldots, \delta_n$ are independent Bernoulli random variables with parameter $1-p$, i.e., $\mathrm{P}(\delta_i = 0) = p$ and $\mathrm{P}(\delta_i = 1) = 1-p$, so the law of $\mathbb{K}_n$ is nonparametric and depends only on p. Moreover, the sequence of empirical processes $\mathbb{K}_n$ converges in law to a continuous centered Gaussian process $\mathbb{K}(t) = W(1-|t|)$, $t \in [-1,1]$, where W is a standard Brownian motion, i.e., a continuous Gaussian process with mean zero and covariance function given by* $\mathrm{cov}\{W(s), W(t)\} = \min(s,t)$, $0 \le s, t \le 1$.

This theorem is useful when performing weighted symmetry tests. Indeed, for any continuous functional Φ on the space of continuous functions $C[-1,1]$, $\Phi(\mathbb{K}_n)$ converges in law to $\Phi(\mathbb{K})$, as $n \to \infty$. In particular, one can take the functionals

$$\Phi(x) = \sup_{-1 \le t \le 1} |x(t)|, \quad x \in C[-1,1]$$

leading to a Kolmogorov–Smirnov statistic, or

$$\Phi(x) = \int_{-1}^{1} \{x(t)\}^2 \, dt,$$

leading to a Cramér–von Mises statistic, or even

$$\Phi(x) = \int_{-1}^{1} x(t) \, dt,$$

leading to a generalized Wilcoxon signed rank statistic which is related to (2).

The following corollary summarizes finite and asymptotic behaviours of these statistics.

COROLLARY 7. *Let $p \in (0,1)$, $\omega > 0$ and $\theta \in \mathbb{R}$ be given constants, and assume that the distribution function F is (p,ω)-symmetric about θ. Then*

(i) *The Kolmogorov–Smirnov statistic*

$$KS_n = \sup_{-1 \le t \le 1} |\mathbb{K}_n(t,p)| = \frac{1}{\sqrt{np(1-p)}} \max_{1 \le j \le n} |S_j|$$

converges in law to $\sup_{0 \le t \le 1} |W(t)|$, *as* $n \to \infty$.

(ii) *The Cramér–von Mises statistic*

$$CvM_n = \frac{1}{2}\int_{-1}^{1} \{\mathbb{K}_n(t,p)\}^2\,dt = \frac{1}{n^2p(1-p)}\sum_{j=1}^{n} S_j^2$$

converges in law to $\int_0^1 \{W(u)\}^2\,du$, *as* $n \to \infty$.

(iii) *The generalized Wilcoxon signed rank statistic* GW_n *defined by*

$$GW_n = -\int_{-1}^{1} \mathbb{K}_n(t,p)\,dt = -\frac{2}{n\sqrt{np(1-p)}}\sum_{j=1}^{n} S_j$$

converges in law, as $n \to \infty$, *to a centered Gaussian random variable with variance* $4/3$ *having representation* $-2\int_0^1 W(u)\,du$.

The generalized Wilcoxon statistic is a normalized version of the statistic AR_n; in fact

$$GW_n = \sqrt{\frac{n}{p(1-p)}}\,\{AR_n - \mathrm{E}\,(AR_n)\}.$$

Note that the law of $\mathbb{K}_n(\,\cdot\,,p)$ is the same as the law of $-\mathbb{K}_n(\,\cdot\,,1-p)$. Since the laws of these three statistics depend only on p, it follows that exact P-values of the associated weighted symmetry tests can be computed for any n. Moreover, when n is large, one can approximate these P-values with the limiting distributions. Indeed, the following relationship (see Shorack & Wellner 1986, p. 34) can be used to approximate the quantiles of KS_n.

$$\mathrm{P}\left\{\sup_{0\le t\le 1}|W(t)| > x\right\} = 1 - \frac{4}{\pi}\sum_{i=0}^{\infty}\frac{(-1)^i}{2i+1}\,e^{-(2i+1)^2\pi^2/8x^2}, \quad x > 0.$$

For instance, the value 2.241 is an approximation to the 95% quantile of KS_n. As can be seen from Table 2, the weak convergence is quite fast.

TABLE 2: 95% quantiles for Kolmogorov–Smirnov statistic KS_n.

n/p	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45	0.50
6	1.780	2.177	1.944	2.041	1.886	1.960	1.926	2.000	1.887	2.041
8	2.596	2.003	1.931	2.121	1.837	2.006	2.075	2.021	2.025	2.121
10	2.322	2.214	2.125	1.897	2.008	2.070	2.122	1.936	2.225	1.897
12	2.185	2.021	2.021	2.021	2.000	2.142	1.967	2.121	2.089	2.021
14	2.023	2.138	2.171	2.138	2.160	2.100	2.157	2.073	2.068	2.138
16	2.007	2.083	2.030	2.000	2.165	2.073	2.149	2.143	2.111	2.000
18	2.271	2.043	1.947	2.121	2.041	2.109	2.125	2.117	2.061	2.121
20	2.206	2.087	2.004	2.124	2.066	2.098	2.133	2.191	2.202	2.236
50	2.109	2.074	2.178	2.121	2.123	2.160	2.194	2.194	2.160	2.121
100	2.134	2.133	2.170	2.200	2.194	2.182	2.201	2.205	2.181	2.200
∞	2.241	2.241	2.241	2.241	2.241	2.241	2.241	2.241	2.241	2.241

Also, P-values associated to the Cramér–von Mises statistic CvM_n can be approximated by means of the following representation (see Orlov 1972)

$$\int_0^1 \{W(u)\}^2\,du \cong \frac{4}{\pi^2}\sum_{k=1}^{\infty}\frac{Z_k^2}{(2k-1)^2},$$

where the Z_k are independent and identically distributed standard Gaussian random variables. By using Table 1 in Shorack & Wellner (1986, p. 748), we get 1.68 as an approximation to the 95% quantile of CvM_n. It follows from Tables 3 and 4 that the weak convergence is quite rapid.

Remark 4. Shorack & Wellner (1986) introduced the rank symmetry process to test for what we refer to as $(p,1)$-symmetry. This process $\mathbb{R}_n$ can be expressed in terms of our process in the following way

$$\mathbb{R}_n(1-t) = 2\sqrt{p(1-p)}\left\{\mathbb{K}_n(t) - \mathbb{K}_n(0) - (p-0.5)(t-[nt]/n)\sqrt{n}\right\}, \quad t \in [0,1].$$

Unfortunately, this process is not centered in general. This causes somewhat imprecise approximation by the limiting process for small and medium sample sizes.

TABLE 3: 95% quantiles for Cramér–von Mises statistic CvM_n.

n/p	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45	0.50
6	2.414	2.441	1.819	1.708	1.731	1.612	1.544	1.685	1.956	1.972
8	1.780	2.056	1.788	1.773	1.646	1.485	1.716	1.773	1.882	1.750
10	2.224	1.828	1.785	1.650	1.577	1.650	1.809	1.792	1.825	1.770
12	2.182	1.782	1.673	1.519	1.505	1.729	1.692	1.690	1.752	1.708
14	2.066	1.755	1.650	1.582	1.726	1.724	1.669	1.679	1.699	1.709
16	1.919	1.630	1.653	1.568	1.688	1.712	1.742	1.721	1.718	1.703
18	1.863	1.697	1.454	1.627	1.692	1.664	1.657	1.737	1.688	1.731
20	1.799	1.619	1.603	1.691	1.678	1.704	1.735	1.688	1.724	1.745
50	1.618	1.652	1.590	1.685	1.679	1.652	1.641	1.748	1.691	1.661
100	1.656	1.622	1.622	1.652	1.639	1.667	1.679	1.654	1.633	1.631
∞	1.68	1.68	1.68	1.68	1.68	1.68	1.68	1.68	1.68	1.68

Remark 5. S-PLUS programs calculating these statistics are also available at the web site http://www.hec.ca/pages/bruno.remillard/Personnel/Splus.

3.2. Parameter p unknown.

Until now, we have assumed that parameters p, ω and θ were known. Hereafter, as a first step, we relax this assumption by assuming that p is unknown, while ω and θ are still given. In this case, a natural estimate of p is the sample proportion of observations $X_i > \theta$, i.e.,

$$\hat{p}_n = \frac{1}{n}\sum_{i=1}^{n} \mathbb{I}(X_i \geq \theta) = 1 - \frac{1}{n}\sum_{i=1}^{n} \delta_i,$$

where the δ_i are defined by (3). Accordingly, one has to modify the weighted symmetry process $\mathbb{K}_n(t,p)$, so we consider its analogue

$$\widetilde{\mathbb{K}}_n(t) = \mathbb{K}_n(t,\hat{p}_n) = \sqrt{\frac{p(1-p)}{\hat{p}_n(1-\hat{p}_n)}}\begin{cases} B_n(1+1/n+[nt]/n), & -1 \leq t < 0, \\ B_n(1-[nt]/n), & 0 \leq t \leq 1, \end{cases}$$

where

$$B_n(u) = W_n(u) - uW_n(1) = \frac{1}{\sqrt{np(1-p)}}\sum_{i=1}^{[nu]}\{\delta_i - (1-k/n)\} = \frac{\widetilde{S}_{[nu]}}{\sqrt{np(1-p)}},$$

provided $\hat{p}_n = k/n$. Though p appears in the above formula, $\widetilde{\mathbb{K}}_n(t)$ does depend on p. In fact, straightforward manipulations show that

$$\widetilde{\mathbb{K}}_n(t) = \frac{1}{\sqrt{n\hat{p}_n(1-\hat{p}_n)}} \sum_{i=1}^{n-[|nt|]} \{\delta_i - (1-\hat{p}_n)\}.$$

Similarly, one can modify the Kolmogorov–Smirnov, Cramér–von Mises, and generalized Wilcoxon signed rank statistics. The analogues of Theorem 6 and Corollary 7 may be stated as follows.

TABLE 4: 95% quantiles for (normalized) Wilcoxon statistic $\sqrt{3/4}\,|GW_n|$.

n/p	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45	0.50
6	2.677	2.318	2.261	2.003	2.109	1.980	1.890	2.021	2.239	2.239
8	2.178	2.143	2.058	2.067	1.945	1.871	2.022	2.125	2.123	2.143
10	2.073	2.100	1.956	1.917	1.929	1.972	2.096	2.124	2.119	2.027
12	2.122	1.972	2.019	1.917	1.876	2.055	2.035	2.058	2.018	2.083
14	2.086	2.039	1.968	1.901	2.004	2.056	1.993	2.025	2.044	2.017
16	2.012	1.931	1.940	1.881	2.000	2.055	2.020	2.022	2.024	2.003
18	2.024	1.958	1.801	1.939	2.030	2.014	1.990	2.056	2.008	2.019
20	1.999	1.936	1.925	1.985	2.035	2.028	2.050	2.016	2.005	2.014
50	1.894	1.951	1.938	1.984	1.977	1.972	1.969	2.030	2.001	1.974
100	1.935	1.934	1.948	1.962	1.978	1.962	1.963	1.976	1.965	1.961
∞	1.96	1.96	1.96	1.96	1.96	1.96	1.96	1.96	1.96	1.96

THEOREM 8. *Let $\omega > 0$ and $\theta \in \mathbb{R}$ be given constants, assume that the distribution function F is (p,ω)-symmetric about θ, with p being unknown. Then the sequence of empirical processes $\widetilde{\mathbb{K}}_n$ converges, as $n \to \infty$, to a $C([-1,1],\mathbb{R})$−valued centered Gaussian process $\widetilde{\mathbb{K}}$, having representation $\widetilde{\mathbb{K}}(t) = B(1-|t|)$, where $B(t) = W(t) - tW(1)$ is called a Brownian bridge. It is a continuous Gaussian process with mean zero and covariance function given by $\operatorname{cov}\{B(s),B(t)\} = \min(s,t) - st$, $0 \le s,t \le 1$. Moreover, conditional on $\hat{p}_n = k/n$, $\delta_1,\ldots\delta_n$ represent successive outcomes of draws without replacement from an urn containing k balls numbered 0 and $n-k$ balls numbered 1, so the law of $\widetilde{\mathbb{K}}_n$ is nonparametric and does not depend on p.*

COROLLARY 9. *Let $\omega > 0$ and $\theta \in \mathbb{R}$ be given constants, and assume that the distribution function F is (p,ω)-symmetric about θ, with p being unknown. Let $\widetilde{S}_0 = 0$, and*

$$\widetilde{S}_j = \sum_{i=1}^{j}\{\delta_i - (1-\hat{p}_n)\}, \quad 1 \le j \le n.$$

Then

(i) *The Kolmogorov–Smirnov statistic*

$$\widetilde{KS}_n = \sup_{-1\le t\le 1} |\widetilde{\mathbb{K}}_n(t)| = \frac{1}{\sqrt{n\hat{p}_n(1-\hat{p}_n)}} \max_{1\le j\le n} |\widetilde{S}_j|$$

converges in law to $\sup_{0\le t\le 1}|B(t)|$, as $n \to \infty$.

(ii) *The Cramér–von Mises statistic*

$$\widetilde{CvM}_n = \frac{1}{2}\int_{-1}^{1}\{\widetilde{\mathbb{K}}_n(t)\}^2\,dt = \frac{1}{n^2\hat{p}_n(1-\hat{p}_n)}\sum_{j=1}^{n}\widetilde{S}_j^2$$

converges in law to $\int_0^1\{B(u)\}^2\,du$*, as* $n\to\infty$*.*

(iii) *The generalized Wilcoxon signed rank statistic* $\widetilde{GW}_n$ *defined by*

$$\widetilde{GW}_n = -\int_{-1}^{1}\widetilde{\mathbb{K}}_n(t)\,dt = -\frac{2}{n\sqrt{n\hat{p}_n(1-\hat{p}_n)}}\sum_{j=1}^{n}\widetilde{S}_j$$

converges in law, as $n\to\infty$*, to a centered Gaussian random variable with variance* $1/3$ *and having representation* $-2\int_0^1 B(u)\,du$*.*

It follows that given $\hat{p}_n$, exact conditional P-values can be computed for any sample sizes, while for large samples, the unconditional P-value can be approximated through the limiting distributions. Indeed, when dealing with the Kolmogorov–Smirnov statistic $\widetilde{KS}_n$, one can rely on the following relationship (see, e.g., Shorack & Wellner 1986, p. 34)

$$\mathrm{P}\left\{\sup_{0\le t\le 1}|B(t)|>x\right\} = 2\sum_{n=1}^{\infty}(-1)^{n+1}e^{-2n^2x^2},\quad x>0.$$

For instance, the associated 95% quantile is 1.358.

Similarly, P-values corresponding to the Cramér–von Mises statistic $\widetilde{CvM}_n$ can be computed from the representation

$$\int_0^1\{B(u)\}^2\,du \cong \frac{1}{\pi^2}\sum_{k=1}^{\infty}\frac{Z_k^2}{k^2},$$

where the Z_k are independent and identically distributed standard Gaussian random variables. A detailed table is given in Shorack & Wellner (1986, p. 147); in particular 0.461 is the 95% quantile.

4. EXAMPLES OF APPLICATIONS

This section is divided into three subsections. The first introduces a kernel estimate for weighted symmetric densities, the second takes a closer look at the gas data, and the third shows that the Toronto stock market index (TSE 300) and the Dow Jones industrial average returns fit weighted symmetric distributions quite nicely.

4.1. Weighted symmetric kernel density estimate.

Consider a probability density f which is (p,ω)-symmetric about θ. For the moment, assume that the parameters p, ω and θ are known. Denote by K a given symmetric and continuous probability density. Define its continuous (though weighted) symmetric version by

$$K(x,\omega) = \begin{cases} 2K(x)/(1+\omega), & x\le 0,\\ 2K(x/\omega)/(1+\omega), & x\ge 0.\end{cases}$$

Given a random sample $X_1,\dots,X_n$ drawn from f, a kernel-based estimate of f can be constructed by setting

$$f_n(\theta+x) = \begin{cases} (1-p)\{g_n(\theta+x)+\omega g_n(\theta-x\omega)\}, & x<0,\\ p\{g_n(\theta+x)+g_n(\theta-x/\omega)/\omega\}, & x\ge 0,\end{cases}$$

where

$$g_n(x) = \frac{1}{nh_n}\sum_{i=1}^{n} K\left(\frac{x - X_i}{h_n}, \omega\right),$$

with h_n being an arbitrary positive smoothing parameter. By construction, the estimator f_n is weighted symmetric about θ, i.e.,

$$pf_n(\theta - x) = (1-p)\omega f_n(\theta + \omega x), \quad x > 0. \tag{6}$$

Since the kernel $K(\,\cdot\,,\omega)$ is continuous in both variables, it follows that $f_n(x)$ is continuous for all $x \neq \theta$. Whereas, at $x = \theta$, one obtains

$$f_n(\theta^-) = (1-p)(1+\omega)g_n(\theta) \quad \text{and} \quad f_n(\theta^+) = p(1+\omega)g_n(\theta)/\omega.$$

Now, one can easily show by the method of Parzen (1962) that the following limit holds with probability one:

$$\begin{aligned}\lim_{n\to\infty} g_n(\theta) &= f(\theta^-)\int_{-\infty}^{0} K(x,\omega)\,dx + f(\theta^+)\int_{0}^{\infty} K(x,\omega)\,dx \\ &= \frac{1}{1+\omega}f(\theta^-) + \frac{\omega}{1+\omega}f(\theta^+).\end{aligned}$$

Consequently, we have with probability one

$$\lim_{n\to\infty} f_n(\theta^-) = f(\theta^-) \quad \text{and} \quad \lim_{n\to\infty} f_n(\theta^+) = f(\theta^+).$$

In summary, the proposed estimator f_n automatically detects the possible discontinuity of the underlying density, in addition to being a bona fide weighted symmetric density.

When the weighted symmetry parameters are unknown, one merely modifies f_n by replacing the unknowns by their estimates discussed in Section 2. A good choice of kernel is given by the following modified Epanechnikov kernel

$$\phi(x,\omega) = \begin{cases} 1.5(1-x^2)/(1+\omega), & -1 \le x \le 0, \\ 1.5(1-x^2/\omega^2)/(1+\omega), & 0 < x \le \omega. \end{cases}$$

Details concerning the selection of a smoothing parameter and consistency of the proposed density estimate will be given in a forthcoming paper.

Finally, to get an idea of the power of the continuity test at θ described in Remark 3, we generated 10,000 split-normal data sets of size $n = 100$ and $n = 250$, with $\theta = 0$ and $p = 1/2$ for each value of $\omega = 0.05, \ldots, 1.95$. Under $\mathcal{H}_0 : \omega/(1+\omega) = p$, one can check that the asymptotic variance is $1/4 + \pi^2/48$. The results of the simulations, as displayed in Figure 3, are quite convincing. The test is very powerful, even for moderate sample sizes like $n = 100$.

In applications, since f is unknown, to estimate the variance, one needs a consistent estimate of the constant c_2, defined in Theorem 4. This is achieved, for example, by setting

$$\hat{c}_2 = \frac{2(1-\hat{p}_n)}{\widehat{\omega}_n\hat{p}_n}\frac{1}{n}\sum_{i;X_i>0} X_i f_n(X_i).$$

4.2. Gas data analysis.

This section is devoted to the analysis of the nonzero differences in gasoline prices for the Québec City area. In Section 1, we stated that the observations were not symmetric about zero. We can now use the results of the previous sections to investigate the weighted symmetry property of that data set. Indeed, the estimation of p, the proportion of positive observations, is given by $\hat{p}_n = 0.3761$, while the Hodges–Lehmann estimate of ω is $\widehat{w}_n = 1.75$.

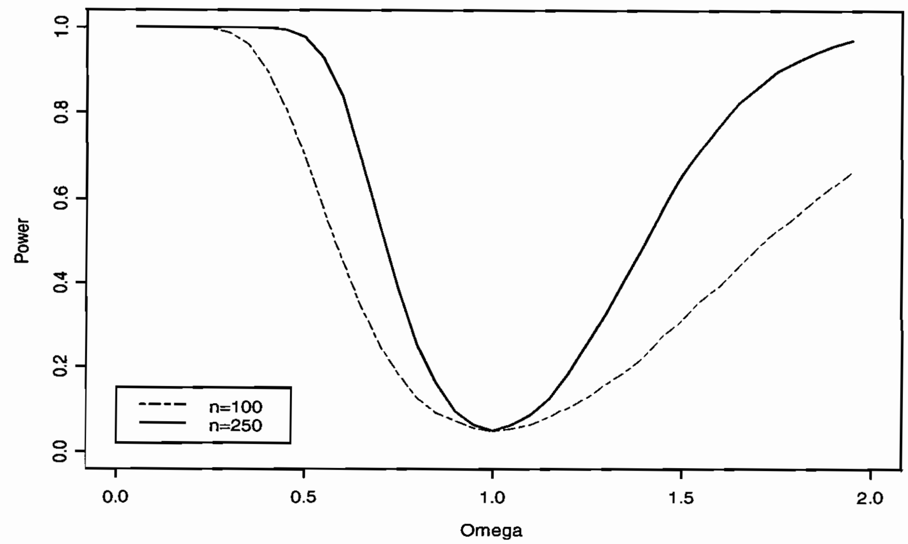

FIGURE 3: Power of the continuity test at the origin under split-normal alternatives.

To check the goodness of fit of the weighted symmetry hypothesis, the data set was randomly split in two halves. The first 54 observations were used to get new parameter estimations $\hat{p}_n = 0.4137$ and $\widehat{w}_n = 1.625$, and the remaining 55 observations were used to calculate the P-values of the three proposed statistics illustrated in Table 5. We calculated the P-values using the asymptotic approximations in Corollary 7 for the distributions of the statistics defined there, treating $\hat{p}_n$ and $\widehat{w}_n$ as known values. These results strongly suggest that the data are weighted symmetric with respect to zero.

Furthermore, a visual assessment of this conclusion is given by the kernel density estimator depicted in Figure 4, which strongly suggests that the density is discontinuous at zero. In fact, continuity at zero is rejected by the test proposed in Remark 3. The value of the statistic is 3.8, yielding a P-value of 0.0001447. In that case, one gets $\hat{c}_2 = 0.0704$. Consequently, one may conclude that the corresponding probability density is discontinuous at zero and that the kernel density depicted in Figure 1 is misleading.

Finally, using the weighted symmetry model, one can see that there are many small decreases in prices compared to relatively few much larger increases in gas prices, in conformance with the consumers' impression of asymmetry in price changes.

TABLE 5: Weighted symmetry tests for the gas data.

Weighted symmetry tests	value	P-value
Generalized Wilcoxon signed rank test	−0.88	0.38
Cramér–von Mises test	0.65	0.25
Kolmogorov–Smirnov test	1.83	0.13

4.3. Stock market returns.

Next consider the monthly (log) returns of the Toronto stock market index (TSE 300) and the Dow Jones industrial average, both for the same period, ranging from January 1985 to April 2002. As expected, classical tests of symmetry confirm that neither return is symmetric about zero. Since zero is a natural cutoff point between positive and negative returns, one expects some kind of weighted symmetry about this point. Indeed, careful analysis shows that both return series are weighted symmetric about zero with parameters $p_n = 0.5922$, $\omega_n = 1.1369$ for the TSE 300 and $p_n = 0.6425$ and $\omega_n = 1.1877$ for the Dow Jones. These findings and the illustration in Figure 5 also show a small but significant difference between the Canadian and American markets, where the latter is often perceived as yielding higher returns. Note that the center of weighted symmetry being zero does not imply that the mean or the median is zero. In fact, as noted earlier, the mean of a (p, ω)-weighted symmetric distribution about zero has the same sign as $p\omega - (1 - p)$. In particular, the mean is positive for both returns.

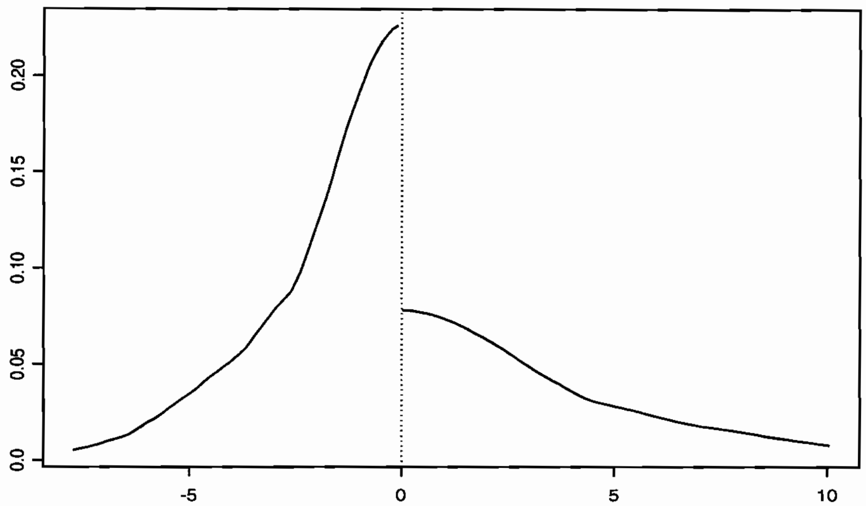

FIGURE 4: A modified kernel density estimate for the gas data.

5. PROOFS

Note that almost all the results given in the previous sections could also be proved by using the powerful machinery of empirical processes for pseudo-observations developed in Ghoudi & Rémillard (1998, 2003). But for the sake of completeness and simplicity, more direct proofs are presented next. Because they are shorter, the proofs of Theorems 6, 8 and Corollaries 7, 9 are given first.

5.1. Proof of Theorem 6.

First recall that from Proposition 1, the ranks of $|X_i - \theta|_\omega$ are independent of the signs $\varepsilon_i = \text{sign}(X_i - \theta)$ and they are uniformly distributed over all possible permutations of the integers $1, \ldots, n$. It follows that $\delta_1, \ldots, \delta_n$ defined by (3) are independent Bernoulli random variables with parameter $1 - p$.

It is well known (see, e.g., Billingsley 1968) that the partial sum process W_n defined in (5), converges in law to a standard Brownian motion W, i.e., a continuous Gaussian process with mean zero and covariance $\text{cov}\{W(s), W(t)\} = \min(s, t)$, $0 \le s, t \le 1$. Therefore, using representation (4), one can conclude that the sequence of processes $\mathbb{K}_n$ converges in law to a continuous Gaussian process $\mathbb{K}_n$ having representation $\mathbb{K}(t) = W(1 - |t|)$, $-1 \le t \le 1$.

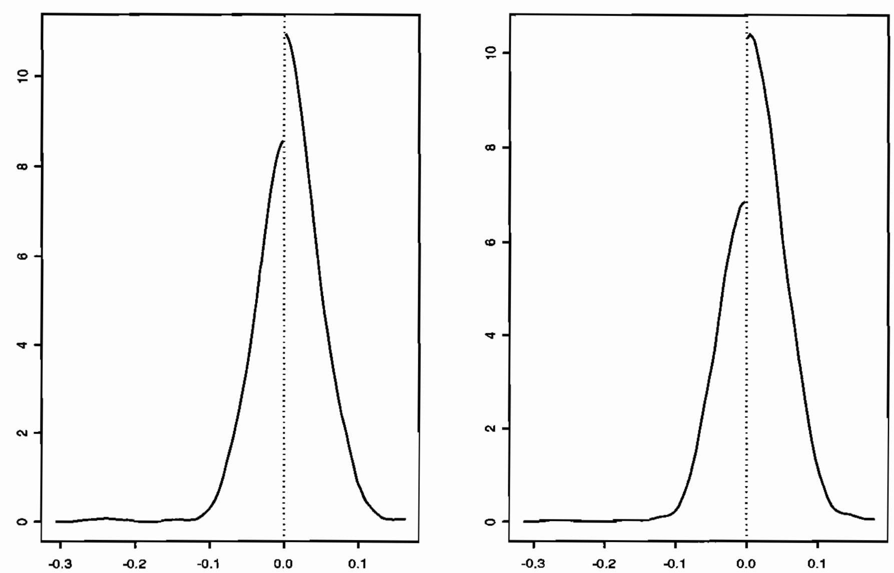

FIGURE 5: A modified kernel density estimate for the stock market indices log returns data: TSE 300 to the left and the Dow Jones to the right.

5.2. Proof of Corollary 7.

The representations of the statistics in terms of the partial sums are obvious from (4) and (5), and their limiting behaviour follows from Donsker's invariance principle and Theorem 6. The only thing to check is the variance of

$$-\int_{-1}^{1} W(1-|t|)\, dt = -2\int_0^1 W(u)\, du.$$

This variance is given by

$$4\int_0^1\int_0^1 \operatorname{cov}\{W(u), W(v)\}\, du\, dv = 4\int_0^1\int_0^1 \min(u,v)\, du\, dv = 4/3.$$

5.3. Proof of Theorem 8.

It follows from Theorem 6 and the representation of $\widetilde{\mathbb{K}}_n$ in terms of W_n, that $\widetilde{\mathbb{K}}_n$ converges to a $B(1-|t|) = W(1-|t|) - (1-|t|)W(1)$. Hence B is a Brownian bridge. Furthermore, under the hypothesis of weighted symmetry, the δ_i are independent Bernoulli random variables with parameter $1-p$. Hence, conditional on $\delta_1 + \cdots + \delta_n = n-k$, the joint law of the δ_i is the same as the law of n draws without replacement from an urn containing k balls numbered 0 and $n-k$ balls numbered 1. Therefore, the conditional law does not depend on p.

5.4. Proof of Corollary 9.

The proof is similar to that of Corollary 7, using the representations in terms of B_n and the partial sums $\widetilde{S}_j$. Finally, the variance of $-2\int_0^1 B(u)\, du$ is given by

$$4\int_0^1\int_0^1 \operatorname{cov}\{B(u), B(v)\}\, du\, dv = 4\int_0^1\int_0^1 \{\min(u,v) - uv\}\, du\, dv = 1/3.$$

5.5. Proof of Theorem 3.

Without loss of generality, it will be assumed that $\theta_0 = 0$. Next, before dealing with the consistency of the Hodges–Lehmann estimate $\hat{\theta}_n$, one needs the asymptotic behaviour of $\sqrt{n}\{AR_n(\theta_n) - AR_n(0)\}$ under the sequence of alternatives $\theta_n = \theta/\sqrt{n}$. In what follows, we shall consider the case $\theta_n > 0$ only; the case $\theta_n < 0$ can be handled in the same manner, and hence is omitted.

Let $H_n(\,\cdot\,, \tau)$ denote the empirical distribution of the $|X_i - \tau|_\omega$ for any $\tau \in \mathbb{R}$. Then, since $\theta_n > 0$, it follows that

$$\begin{aligned}
\sqrt{n}\{AR_n(\theta_n) - AR_n(0)\} &= \frac{1}{\sqrt{n}} \sum_{i=1}^n \{\text{sign}(X_i - \theta_n) H_n(|X_i - \theta_n|_\omega, \theta_n) \\
&\quad - \text{sign}(X_i) H_n(|X_i|_\omega, 0)\} \\
&= \frac{1}{\sqrt{n}} \sum_{i=1}^n \mathbb{I}(X_i > \theta_n)\{H_n(X_i - \theta_n, \theta_n) - H_n(X_i, 0)\} \\
&\quad - \frac{1}{\sqrt{n}} \sum_{i=1}^n \mathbb{I}(0 < X_i \le \theta_n)\{H_n(-\omega(X_i - \theta_n), \theta_n) + H_n(X_i)\} \\
&\quad - \frac{1}{\sqrt{n}} \sum_{i=1}^n \mathbb{I}(X_i \le 0)\{H_n(-\omega(X_i - \theta_n), \theta_n) - H_n(-\omega X_i)\} \\
&\equiv T_1 - T_2 - T_3.
\end{aligned}$$

The consistency of each of these terms is considered in turn. Let $o_p(1)$ denote any sequence of random variables converging in probability to zero. Set $\beta_n(t) = \sqrt{n}\{F_n(t) - F(t)\}$, with F_n standing for the empirical distribution of the X_i, and note that for any $t \ge 0$ and any $\tau \in \mathbb{R}$, one has $H_n(t, \tau) = F_n(t + \tau) - F_n(\tau - t/\omega^-)$. Therefore,

$$T_1 = \frac{1}{\sqrt{n}} \sum_{i=1}^n \mathbb{I}(X_i > \theta_n)\Big[F_n(-X_i/\omega^-) - F_n\big[\{\theta_n(\omega+1) - X_i\}/\omega^-\big]\Big] \equiv T_{11} + T_{12},$$

where

$$\begin{aligned}
T_{11} &= \frac{1}{n} \sum_{i=1}^n \mathbb{I}(X_i > \theta_n)\Big[\beta_n(-X_i/\omega^-) - \beta_n\big[\{\theta_n(\omega+1) - X_i\}/\omega^-\big]\Big], \\
T_{12} = &= \frac{1}{\sqrt{n}} \sum_{i=1}^n \mathbb{I}(X_i > \theta_n)\Big[F(-X_i/\omega) - F\big[\{\theta_n(\omega+1) - X_i\}/\omega\big]\Big].
\end{aligned}$$

As for the second term T_2, note that

$$\mathrm{E}\left[\sum_{i=1}^n \{F_n(X_i) - F_n(X_i^-)\}\right] = 1$$

because F is continuous, so one has the following chain of inequalities:

$$\begin{aligned}
T_2 &= \frac{1}{\sqrt{n}} \sum_{i=1}^n \mathbb{I}(0 < X_i \le \theta_n)\big[F_n\{\theta_n(1+\omega) - \omega X_i\} - F_n(X_i^-)\big] \\
&\le \frac{1}{\sqrt{n}} \sum_{i=1}^n \mathbb{I}(0 < X_i \le \theta_n)\big[F_n\{\theta_n(1+\omega)\} - F_n(-\theta_n/\omega^-)\big] + o_p(1)
\end{aligned}$$

$$\begin{aligned}
&\leq \{F_n(\theta_n) - F_n(0)\}\left[\beta_n\{\theta_n(1+\omega)\} - \beta_n(-\theta_n/\omega^-)\right] \\
&\quad + \{\beta_n(\theta_n) - \beta_n(0)\}\left[F\{\theta_n(1+\omega)\} - F(-\theta_n/\omega)\right] \\
&\quad + \sqrt{n}\left[F\{\theta_n(1+\omega)\} - F(-\theta_n/\omega)\right]^2 + o_p(1) \\
&\equiv T_{21} + T_{22} + T_{23} + o_p(1).
\end{aligned}$$

Finally, the third term T_3 satisfies

$$\begin{aligned}
T_3 &= \frac{1}{\sqrt{n}}\sum_{i=1}^n \mathbb{I}(X_i \leq 0)\left[F_n\{\theta_n(1+\omega) - \omega X_i\} - F_n(-\omega X_i)\right] \\
&= \frac{1}{n}\sum_{i=1}^n \mathbb{I}(X_i \leq 0)\left[\beta_n\{\theta_n(1+\omega) - \omega X_i\} - \beta_n(-\omega X_i)\right] \\
&\quad + \frac{1}{\sqrt{n}}\sum_{i=1}^n \mathbb{I}(X_i \leq 0)\left[F\{\theta_n(1+\omega) - \omega X_i\} - F(-\omega X_i)\right] \\
&\equiv T_{31} + T_{32}.
\end{aligned}$$

Now, let α_n be the empirical process constructed with the $F(X_i)$ and rewrite $\beta_n = \alpha_n \circ F$. Then, α_n being tight and F being continuous, it follows that all the terms T_{11}, T_{21}, T_{22} and T_{31} converge to zero in probability. Furthermore, one has the following inequality

$$T_{23} \leq \theta\left(1 + \omega + \frac{1}{\omega}\right)\int_{-\theta_n/\omega}^{\theta_n(1+\omega)} f^2(u)\,du.$$

The last integral converges to zero as $n \to \infty$ because the density f is square integrable. Using these consistency results together with the three decompositions above, one can write

$$\sqrt{n}\,\{AR_n(\theta_n) - AR_n(0)\} = T_{12} - T_{32} + o_p(1).$$

Next, as a by-product of the preceding calculations, $T_{12} - T_{32} = M_n + o_p(1)$, where

$$\begin{aligned}
M_n &= \frac{1}{\sqrt{n}}\sum_{i=1}^n \mathbb{I}(X_i > 0)\Big[F(-X_i/\omega) - F\left[\{\theta_n(\omega+1) - X_i\}/\omega\right]\Big] \\
&\quad - \frac{1}{\sqrt{n}}\sum_{i=1}^n \mathbb{I}(X_i \leq 0)\left[F\{\theta_n(1+\omega) - \omega X_i\} - F(-\omega X_i)\right].
\end{aligned}$$

A straightforward application of the weak law of large numbers shows that $M_n - \mathrm{E}(M_n)$ converges in probability to zero, while

$$\mathrm{E}(M_n) = -\int_{-\infty}^{\infty}\int_0^{\theta(1+\omega)/\omega} h_{1n}(z,u)f(z)\,du\,dz - \int_{-\infty}^{\infty}\int_0^{\theta(1+\omega)} h_{2n}(z,u)f(z)\,du\,dz,$$

where

$$h_{1n}(z,u) = f\left(-\frac{z}{\omega} + \frac{u}{\sqrt{n}}\right)\mathbb{I}(z > 0) \quad \text{and} \quad h_{2n}(z,u) = f\left(-\omega z + \frac{u}{\sqrt{n}}\right)\mathbb{I}(z \leq 0).$$

Since f is square integrable, the mapping

$$v \mapsto \int_0^{\infty}\{f(v-z) - f(-z)\}^2\,dz$$

is continuous, because, by Theorem 9.5 of Rudin (1987), the mapping $y \mapsto g(\cdot + y)$ is continuous in L^2 for any square integrable function g.

Hence the dominated convergence theorem entails that

$$\lim_{n\to\infty} \mathrm{E}\,(M_n) = -2\theta(1+\omega)\int_0^\infty f(\omega z)f(-z)\,dz.$$

Next, by virtue of the weighted symmetry assumption, the density f satisfies

$$pf(-z) = (1-p)\omega f(\omega z)$$

for almost every $z > 0$. Consequently, by gathering the previous results, one ends up with

$$\lim_{n\to\infty} \sqrt{n}\,\{AR_n(\theta_n) - AR_n(0)\} = -c_1\theta, \tag{7}$$

in probability, where

$$c_1 = 2(1+\omega)\frac{1-p}{p}\int_0^\infty f^2(z)\,dz.$$

Finally, set $\mu_n = (n+1)(2p-1)/(2n)$. Since $\sqrt{n}\,\{AR_n(0)-\mu_n\}\,/\,\sqrt{p(1-p)}$ converges in law to a centered Gaussian random variable with variance $4/3$, it follows easily that $\sqrt{n}\,(\hat{\theta}_n - \theta_0)$ converges in law to a centered Gaussian random variable with variance $4/(3c_1^2)$. This argument is relatively standard for Hodges–Lehmann estimators, so it is omitted.

5.6. Proof of Theorem 4.

The proof of this theorem follows the same lines as that of Theorem 3. Without loss of generality, one can assume that $\theta = 0$. Set

$$H_n(t,\omega) = \frac{1}{n}\sum_{i=1}^{n} \mathbb{I}(|X_i|_\omega \le t)$$

and

$$AR_n(\omega) = \frac{1}{n}\sum_{i=1}^{n} \mathrm{sign}\,(X_i) H_n(|X_i|_\omega, \omega),$$

for any $\omega > 0$. Here again, one needs the asymptotic behaviour of $\sqrt{n}\,\{AR_n(\omega_n) - AR_n(\omega_0)\}$ for contiguous alternatives of the form $\omega_n = \omega_0 + \omega/\sqrt{n}$. Standard algebra yields

$$\begin{aligned}
\sqrt{n}\,\{AR_n(\omega_n) - AR_n(\omega_0)\} &= \frac{1}{\sqrt{n}}\sum_{i=1}^{n} \mathrm{sign}(X_i)\{H_n(|X_i|_{\omega_n}, \omega_n) - H_n(|X_i|_{\omega_0}, \omega_0)\} \\
&= \frac{1}{\sqrt{n}}\sum_{i=1}^{n} \mathbb{I}(X_i > 0)\{F_n(-X_i/\omega_0^-) - F_n(-X_i/\omega_n^-)\} \\
&\quad + \frac{1}{\sqrt{n}}\sum_{i=1}^{n} \mathbb{I}(X_i \le 0)\{F_n(-\omega_0 X_i) - F_n(-\omega_n X_i)\} \\
&= R_1 + R_2 + R_3 + R_4,
\end{aligned}$$

where

$$\begin{aligned}
R_1 &= \frac{1}{n}\sum_{i=1}^{n} \mathbb{I}(X_i > 0)\{\beta_n(-X_i/\omega_0^-) - \beta_n(-X_i/\omega_n^-)\}, \\
R_2 &= \frac{1}{n}\sum_{i=1}^{n} \mathbb{I}(X_i \le 0)\{\beta_n(-\omega_0 X_i) - \beta_n(-\omega_n X_i)\}, \\
R_3 &= \frac{1}{\sqrt{n}}\sum_{i=1}^{n} \mathbb{I}(X_i > 0)\{F(-X_i/\omega_0) - F(-X_i/\omega_n)\}, \\
R_4 &= \frac{1}{\sqrt{n}}\sum_{i=1}^{n} \mathbb{I}(X_i \le 0)\{F(-\omega_0 X_i) - F(-\omega_n X_i)\},
\end{aligned}$$

and where, once again, $\beta_n(t) = \sqrt{n}\,\{F_n(t) - F(t)\} = \alpha_n\{F(t)\}$, with α_n being the empirical process constructed from the $F(X_i)$. The tightness of α_n entails that both R_1 and R_2 converge in probability to zero, while an application of the weak law of large number gives

$$\lim_{n\to\infty} R_3 + R_4 - \mathrm{E}\,(R_3 + R_4) = 0,$$

in probability. Thus, it remains to investigate the remaining term $\mathrm{E}\,(R_3 + R_4)$. One has

$$\begin{aligned}\mathrm{E}\,(R_3 + R_4) &= -\frac{1}{\omega_n}\int_0^\infty\int_0^{\omega/\omega_0} zf(z)f\big\{zu/(\omega_n\sqrt{n}\,) - z/\omega_0\big\}\,du\,dz\\ &\quad - \omega_0\int_0^\infty\int_0^{\omega/\omega_0} zf(-z)f(\omega_0 zu/\sqrt{n} + \omega_0 z)\,du\,dz.\end{aligned}$$

Since $|z|f^2(z)$ is integrable, it follows from Theorem 3.14 in Rudin (1987) that the mapping

$$u \mapsto \int_{-\infty}^\infty |z|\{f(zu) - f(z)\}^2\,dz$$

is continuous, because any integrable function can be approximated in L^1 by continuous functions of compact support. Hence $\mathrm{E}\,(R_3 + R_4)$ converges to

$$-2\omega\int_0^\infty zf(-z)f(\omega_0 z)\,dz.$$

Consequently, by using the weighted symmetry assumption one gets

$$\lim_{n\to\infty}\sqrt{n}\,\{AR_n(\omega_n) - AR_n(\omega_0)\} = -c_2\omega$$

in probability, where

$$c_2 = \frac{2}{\omega_0}\frac{1-p}{p}\int_0^\infty zf^2(z)\,dz.$$

Set $\mu_n = (n+1)(2q_n - 1)/(2n)$. Using the same arguments as the ones used at the end of the proof of Theorem 3, the result follows since $\sqrt{n}\,\{AR_n(\omega_0) - \mu_n\}\ /\ \sqrt{q_n(1-q_n)}$ converges in law to a centered Gaussian random variable with variance $\sigma^2 = 4/3$ or $1/3$, according as p is known or not.

Finally, when $q_n = \hat{p}_n$, it is easy to check that $\sqrt{n}\,\{AR_n(\omega_0) - \mu_n\}$ and $\sqrt{n}\,(\hat{p}_n - p)$ converge jointly to independent Gaussian variables. Hence $\sqrt{n}\,(\widehat{\omega}_n - \omega_0)$ and $\sqrt{n}\,(\hat{p}_n - p)$ also converge jointly to independent Gaussian variables.

5.7. Proof of Theorem 5.

The proof mimics the same steps as the proof of Blackman's Theorem given in Shorack & Wellner (1986, p. 254). Only the key points are outlined next. First notice that

$$\begin{aligned}T_n(\theta,\omega,p) &= \int_0^\infty \Big[p\alpha_n(\theta - x^-) + (1-p)\alpha_n(\theta + \omega x)\\ &\qquad + p\sqrt{n}\,F(\theta - x) - (1-p)\sqrt{n}\,\{1 - F(\theta + \omega x)\}\Big]^2\,dx,\end{aligned}$$

where α_n is the empirical process constructed from $X_1,\ldots,X_n$.

Following Shorack & Wellner (1986, see pp. 759, 254 and the references therein), one sees that

$$\mathrm{E}\left[\sup_{\|(a,b,c)\|\le B}\Big|T_n\big(\theta_0 + a/\sqrt{n}\,, \omega_0 + b/\sqrt{n}\,, p_0 + c/\sqrt{n}\,\big) - M(a,b,c)\Big|\right] \to 0,$$

where

$$\begin{aligned} M(a,b,c) &= \int_0^\infty \Big[p_0\alpha(\theta_0 - x) + (1-p_0)\alpha(\theta_0 + \omega_0 x) \\ &\quad + c\{F(\theta_0 - x) + 1 - F(\theta_0 + \omega_0 x)\} \\ &\quad + a\{p_0 f(\theta_0 - x) + (1-p_0)f(\theta_0 + \omega_0 x)\} + b(1-p_0)xf(\theta_0 + \omega_0 x)\Big]^2 dx, \end{aligned}$$

with α being the limit of the empirical process α_n. Note that M is a quadratic form in (a, b, c). For $\|(a, b, c)\| > B$, one mimics the arguments in the above references and shows that the minimum of T_n is not in this region (i.e., T_n is quite large in this region). Solving for the minimum of M, one concludes that $(\sqrt{n}\,(\hat\theta - \theta_0), \sqrt{n}\,(\widehat\omega - \omega_0), \sqrt{n}\,(\hat p - p_0))$ converges to a multivariate normal distribution given by $A^{-1}V$, where

$$V = -\begin{pmatrix} \int_0^\infty \{p_0\alpha(\theta_0 - x) + (1-p_0)\alpha(\theta_0 + \omega_0 x)\} h_1(x, \theta_0, \omega_0, p_0)\, dx \\ \int_0^\infty \{p_0\alpha(\theta_0 - x) + (1-p_0)\alpha(\theta_0 + \omega_0 x)\} h_2(x, \theta_0, \omega_0, p_0)\, dx \\ \int_0^\infty \{p_0\alpha(\theta_0 - x) + (1-p_0)\alpha(\theta_0 + \omega_0 x)\} h_3(x, \theta_0, \omega_0, p_0)\, dx. \end{pmatrix}$$

Easy calculations show that the process $p_0\alpha(\theta_0 - x) + (1-p_0)\alpha(\theta_0 + \omega_0 x)$ is a mean zero Gaussian process with covariance function given by $\Gamma_1(x, y) = p_0 \min\{F(\theta_0 - x), F(\theta_0 - y)\}$.

5.8. Proof of Proposition 2.

By hypothesis, F is continuous. First notice that if F is weighted symmetric about θ, then either f is discontinuous at θ, or else f is continuous at θ, in which case

$$f(\theta - x) = \omega(1-p)p^{-1} f(\theta + \omega x) \ \text{ and } \ \{f(\theta - x) - f(\theta)\}\{f(\theta + \omega x) - f(\theta)\} \geq 0$$

hold in a neighbourhood of θ. Moreover, if f is continuous at θ and $f(\theta) \neq 0$, then $(1-p)\omega/p = 1$.

Next we assume that F is weighted symmetric about two points $\theta_1 < \theta_2$, and we show that this contradicts our hypothesis. Let p_1, ω_1 be associated to θ_1 and p_2, ω_2 be associated to θ_2. Observe that $0 < p_2 \leq p_1 < 1$.

Let $b_0 = \theta_1$, let $A_0 = \theta_2$, and for $n \geq 1$, let

$$\begin{aligned} a_n &= \theta_2 + \omega_2(\theta_2 - b_{n-1}), \qquad b_n = \theta_1 - (a_n - \theta_1)/\omega_1, \\ B_n &= \theta_1 - (A_{n-1} - \theta_1)/\omega_1, \quad A_n = \theta_2 + \omega_2(\theta_2 - B_n). \end{aligned}$$

Observe that a_n and A_n are strictly increasing sequences and b_n and B_n are strictly decreasing sequences. Also note that the shape of f around a_n is a copy of the shape of f around θ_1 and the shape of f around A_n is a copy of the shape around θ_2.

Set

$$r = \frac{p_2(1-p_1)}{p_1(1-p_2)} \quad \text{and} \quad \Delta = \frac{\omega_2}{\omega_1}.$$

Easy computations yield

$$a_{n+1} - a_n = \omega_2(b_{n-1} - b_n) = \Delta(a_n - a_{n-1}),$$

$$b_{n+1} - b_n = \frac{a_n - a_{n+1}}{w_1} = \Delta(b_n - b_{n-1}),$$

$$A_{n+1} - A_n = \Delta(A_n - A_{n-1}),$$

$$B_{n+1} - B_n = \Delta(B_n - B_{n-1}),$$

and

$$
\begin{aligned}
F(a_{n+1}) - F(a_n) &= \frac{p_2}{1-p_2}\{F(b_{n-1}) - F(b_n)\} = r\{F(a_n) - F(a_{n-1})\},\\
F(b_{n+1}) - F(b_n) &= \frac{1-p_1}{p_1}\{F(a_{n-1}) - F(a_n)\} = r\{F(b_n) - F(b_{n-1})\},\\
F(A_{n+1}) - F(A_n) &= r\{F(A_n) - F(A_{n-1})\},\\
F(B_{n+1}) - F(B_n) &= r\{F(B_n) - F(B_{n-1})\}.
\end{aligned}
$$

One also has

$$
F(b_0) - F(b_1) = (1-r)(1-p_1), \quad F(A_1) - F(A_0) = (1-r)p_2.
$$

One first shows that $r < 1$, i.e., $p_1 > p_2$. To prove that, assume the contrary, i.e., $p_1 = p_2$, since $p_1 \geq p_2$. Then $F(b_n) = 1 - p_1$ for all $n \geq 0$. It follows that $\omega_2 < \omega_1$. Otherwise, $b_n \to -\infty$, so $1 - p_1 = F(b_n) \to 0$, which is impossible. Next set $\underline{L} = \inf\{x : F(x) = 1 - p_1\}$ and $\overline{L} = \sup\{x : F(x) = 1 - p_1\}$.

Since $p_1 = p_2$, for $i = 1, 2$,

$$
\begin{aligned}
1 - p_1 = F(\underline{L}) &= F\{\theta_i - (\theta_i - \underline{L})\} = \frac{1-p_i}{p_i}[1 - F\{\theta_i + \omega_2(\theta_i - \underline{L})\}]\\
&= \frac{1-p_1}{p_1}[1 - F\{\theta_i + \omega_i(\theta_i - \underline{L})\}],
\end{aligned}
$$

so $F\{\theta_i + \omega_i(\theta_i - \underline{L})\} = 1 - p_1$ and hence $\overline{L} \geq \theta_i + \omega_i(\theta_i - \underline{L})$, $i = 1, 2$.

On the other hand, for $i = 1, 2$,

$$
\begin{aligned}
p_1 = 1 - F(\overline{L}) &= 1 - F\{\theta_i + (\overline{L} - \theta_i)\} = \frac{p_i}{1-p_i}F\{\theta_i - (\overline{L} - \theta_i)/\omega_i\}\\
&= \frac{p_1}{1-p_1}F\{\theta_i - (\overline{L} - \theta_i)/\omega_i\},
\end{aligned}
$$

so $F\{\theta_i - (\overline{L} - \theta_i)/\omega_i\} = 1 - p_1$ and hence $\underline{L} \leq \theta_i - (\overline{L} - \theta_i)/\omega_i)$, which is equivalent to $\overline{L} \leq \theta_i + \omega_i(\theta_i - \underline{L})$, $i = 1, 2$.

Combining these inequalities, one can conclude that $\overline{L} = \theta_i + \omega_i(\theta_i - \underline{L})$, $i = 1, 2$, so

$$
\underline{L} = \frac{(1+\omega_1)\theta_1 - (1+\omega_2)\theta_2}{\omega_1 - \omega_2}.
$$

Next, since $\theta_2 + \omega_2 x = \theta_1 + \omega_1 x'$ yields

$$
F(\theta_2 - x) = F(\theta_1 - \omega_1 x') = F\left\{\theta_1 - \frac{(\theta_2 - \theta_1)}{\omega_1} - \frac{\omega_2}{\omega_1}x\right\}
$$

for all $x \geq 0$, it follows that for all $x \leq \theta_2$,

$$
F(x) = F\left\{\underline{L} + \frac{\omega_2}{\omega_1}(x - \underline{L})\right\}.
$$

Let $x < \underline{L}$ be fixed. Then $x < x_1 = \underline{L} + \omega_2(x - \underline{L})/\omega_1 < \underline{L}$ and $F(x) = F(x_1)$. Next $x_1 < x_2 = \underline{L} + \omega_2(x_1 - \underline{L})/\omega_1 = \underline{L} + (\omega_2/\omega_1)^2(x - \underline{L}) < \underline{L} < \underline{L}$ and $F(x) = F(x_2)$. Defining $x_n = \underline{L} + (\omega_2/\omega_1)^n(x_n - \underline{L})$, it follows that $x_n < \underline{L}$, $x_n \to \underline{L}$ and $F(x) = F(x_n)$. By continuity, $F(x_n) \to F(\underline{L})$, so $F(x) = F(\underline{L}) = 1 - p_1$, which is absurd, since $\underline{L}$ is the infimum. Therefore, one must conclude that $p_1 > p_2$, so $p_2(1-p_1) < p_1(1-p_2)$ and hence $r < 1$.

Set $b_\infty = \lim_{n\to\infty} b_n$. Then

$$1 - p_1 - F(b_\infty) = \sum_{i=1}^{\infty} F(b_{i-1}) - F(b_i) = (1-p_1)(1-r)\sum_{n=0}^{\infty} r^n = 1 - p_1,$$

and it follows that $F(b_\infty) = 0$. Moreover, since $F(b_n) - F(b_{n+1}) > 0$ for all $n \geq 0$, one can see that $[b_{n+1}, b_n]$ has nonempty intersection with the support of F.

Similarly, setting $A_\infty = \lim_{n\to\infty} A_n$, one obtains

$$F(A_\infty) - F(\theta_2) = \sum_{n=0}^{\infty} F(A_{n+1}) - F(A_n) = (1-r)p_2 \sum_{n=0}^{\infty} r^n = p_2,$$

so $F(A_\infty) = 1$. Since $F(A_{n+1}) - F(A_n) > 0$ for all $n \geq 0$, it follows that $[A_n, A_{n+1}]$ has nonempty intersection with the support of F.

Next, for all $x \geq 0$,

$$\begin{aligned} F(\theta_1 - x) &= F\{\theta_2 - (\theta_2 - \theta_1 + x)\} \\ &= \frac{1-p_2}{p_2}\big[1 - F\{\theta_2 + \omega_2(\theta_2 - \theta_1 + x)\}\big] \\ &= \frac{1-p_2}{p_2}\big[1 - F\{\theta_1 + (\theta_2 - \theta_1)(1+\omega_2) + \omega_2 x\}\big] \\ &= \frac{1}{r} F\{\theta_1 - (\theta_2 - \theta_1)(1+\omega_2)/\omega_1 - \Delta x\} \\ &= \frac{1}{r} F(b_1 - \Delta x). \end{aligned}$$

Similarly, for all $0 \leq x \leq \theta_2 - \theta_1$,

$$F(\theta_1 + x) = F\{\theta_2 - (\theta_2 - \theta_1 - x)\} = \frac{1}{r} F(b_1 + \Delta x).$$

By iteration, one can check that for all x small enough, $r^n F(\theta_1 + x) = F(b_n + \Delta^n x)$. One can also see that for all x small enough, $r^n F(\theta_2 + x) = F(A_n + \Delta^n x)$. Therefore, for all $n \geq 1$,

$$r^n f(\theta_1^{\pm}) = \Delta^n f(b_n^{\pm}) \quad \text{and} \quad r^n f(\theta_2^{\pm}) = \Delta^n f(A_n^{\pm}).$$

To complete the proof we distinguish three cases.

Case 1: f is discontinuous at θ_1 or θ_2. If f is discontinuous at θ_1 (respectively θ_2), then by the above f will be discontinuous at b_n (respectively A_n) for all $n \geq 1$, i.e., f has an infinite number or discontinuities, contradicting our hypothesis.

Case 2: $f(\theta_1) = 0$ or $f(\theta_2) = 0$. Assume $f(\theta_1) = 0$. This yields $f(b_n) = 0$ for all $n \geq 1$. Either f is continuous on $[b_{n-1}, b_n]$, so it has a nonzero maximum on the interval, or f has a discontinuity in the interval. Therefore, f has an infinite number of discontinuities or an infinite number of local maxima in its support, which contradicts our hypothesis. The same reasoning applies if $f(\theta_2) = 0$.

Case 3: f is continuous at θ_1 and θ_2, $f(\theta_1) > 0$ and $f(\theta_2) > 0$. It follows that $\Delta = r < 1$, so $f(\theta_1) = f(b_n)$ for all $n \geq 1$. That is, for all $n \geq 1$, one has $f(b_{n-1}) = f(b_n) = f(\theta_1)$. So either f has a discontinuity in the interval $[b_n, b_{n-1}]$ or f is continuous in this interval, in which case its maximum is greater or equal to $f(\theta_1) > 0$. Both situations contradict our hypothesis.

So for any of the three cases, if F satisfies the conditions of Proposition 2, then $\theta_1 = \theta_2$ which implies that $p_1 = 1 - F(\theta_1) = p_2$. It remains to prove that $\omega_1 = \omega_2$. Suppose the contrary and set $\Delta' = \min(\omega_1, \omega_2)/\max(\omega_1, \omega_2) < 1$. Since $F(\theta - x/\omega_1) = F(\theta - x/\omega_2)$ for all $x \geq 0$, it follows that for all $y \leq \theta$,

$$F(y) = F\{(1 - \Delta')\theta + \Delta' y\}.$$

Let $y < \theta$ be given and set $x_0 = y$, $x_n = (1 - \Delta')\theta + \Delta' x_{n-1}$, $n \geq 1$. Then x_n is a strictly increasing sequence, $x_n < \theta$, $F(x_n) = F(y)$ and

$$\theta - x_n = \Delta'(\theta - x_{n-1}) = (\Delta')^n(\theta - y), \quad n \geq 1.$$

It follows that $x_n \to \theta$, so by continuity,

$$F(\theta) = \lim_{n\to\infty} F(x_n) = F(y).$$

Therefore $F(y) = F(\theta)$ for all $y \leq \theta$, which is impossible since $F(\theta) = 1 - p > 0$. Consequently, $\Delta' < 1$ leads to a contradiction and one can conclude that $\omega_1 = \omega_2$.

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