Differential Equations and Engineering Applications

My Students \(^1\)

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\(^1\)It is mostly based on the textbook, Peter V. O’Neil, *Advanced Engineering Mathematics*, 5th Edition, and has been reorganized and retyped by Jae Lee
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A differential equation is an equation that contains one or more derivatives. For example,
\[ y''(x) + x^3 y'(x) + y^5(x) = 10 \cos(4x), \quad \frac{d^4w}{dt^4} - (w(t))^3 = e^{-5t}. \]

If the differential equation is involved with only total derivatives, then it is called an ordinary differential equation. If it contains the partial derivatives, then we call it a partial differential equation. In this chapter, we study only the ordinary differential equation.

The order of a differential equation is the order of its highest derivative. For instance, the differential equation \( xy' - y^3 = e^x \) has the first–order derivative, which is the highest derivative, and so it is the first–order differential equation. The differential equation \( y^4y'' - xy' + y^3 = e^x \) has the second–order derivative and so it is the second–order differential equation. In this chapter, we study only the first–order differential equation and in this course, we lay our concern on only the first–order and second–order differential equations.

The solution of a differential equation is any function that satisfies it. For example, \( y = \cos(3x) \) is a solution of the second–order differential equation \( y'' + 9y = 0 \).

**Proof.** Differentiating \( y = \cos(3x) \) twice, we get
\[ y' = -3 \sin(3x), \quad y'' = -9 \cos(3x). \]

Putting it into the given equation, we observe
\[ y'' + 9y = -9 \cos(3x) + 9 \cos(3x) = 0, \]

i.e., \( y = \cos(3x) \) satisfies the given differential equation \( y'' + 9y = 0 \), hence it is a solution of the given differential equation.

A solution may be defined on the entire real line or on only part of the real line, often an interval. For example, \( y = x \ln x - x \) is a solution of \( y' = y/x + 1 \). Obviously the solution \( y = x \ln x - x \) is defined for \( x > 0 \), because \( \ln x \) has the domain \( x > 0 \).

§1.1 Preliminary Concepts.

**□ 1.1.1 General and Particular Solutions.**

Recall that a first–order differential equation is an equation involving a first derivative. The general form of a first–order differential equation is given by

\[ F(x, y, y') = 0, \quad (1.1.1) \]

where \( y \) is the function of \( x \) and \( y' \) is the derivative of \( y \) with respect to \( x \).

**Example 1.1.1.** The followings are first–order differential equations.

\[ y' - y^2 - e^y = 0, \quad y' - 2 = 0, \quad y' - \cos x = 0, \quad x^5y' + y^{100} - \ln(xy^4) = 0. \]

A solution of equation \((1.1.1)\) on an interval \( I \) is a function \( \varphi(x) \) satisfying the equation for all \( x \) in \( I \). That is, \( \varphi(x) \) is a solution of equation \((1.1.1)\) on the interval \( I \) if and only if \( F(x, \varphi(x), \varphi'(x)) = 0 \) for all \( x \) in \( I \).

**Example 1.1.2.** The function \( \varphi(x) = 2 + ke^{-x} \) is a solution of the differential equation \( y' + y = 2 \) for all real \( x \) and for any number \( k \).
Proof. Differentiating $\varphi(x)$ and putting it into the given equation, we observe

$$\varphi'(x) = -ke^{-x}, \quad \varphi'(x) + \varphi(x) = -ke^{-x} + 2 + ke^{-x} = 2,$$

i.e., $\varphi'(x) + \varphi(x) = 2$, that is, the function $\varphi(x) = 2 + ke^{-x}$ satisfies the given equation, hence it is a solution of the equation. \[\Box\]

In this example, as we can see, the solution contains an arbitrary constant $k$. Such a solution is called a **general solution** of the differential equation. That is, $\varphi(x) = 2 + ke^{-x}$ is the general solution of $y' + y = 2$. Each choice of the constant in the general solution gives a **particular solution**. For example, $\varphi(x) = 2 + e^{-x}$ and $\varphi(x) = 2 - 10e^{-x}$ also satisfy the equation $y' + y = 2$ and so they are particular solutions of the equation.

As one can see, those particular solutions can be obtained by putting $k = 1$ and $k = -10$ into the general solution.

One of the main goals in this course is to find the general solution of a differential equation. We will develop the techniques/methods to find the general solutions of the first–order and second–order differential equations in the later chapters.

\[\Box\] 1.1.2 Implicitly Defined Solutions.

**Example 1.1.3.** The differential equation $y' = -y$ has the general solution $y = ke^{-x}$.

**Example 1.1.4.** The differential equation

$$y' = -\frac{2xy^3 + 2}{3x^2y^2 + 8e^{4y}}$$

has the general solution $y$ implicitly defined by the equation

$$x^2y^3 + 2x + 2e^{4y} = k,$$

where $k$ is an arbitrary constant.

In the Example 1.1.3, the solution is explicitly obtained: $y = ke^{-x}$. However, in the Example 1.1.4, the solution is implicitly obtained by the equation $x^2y^3 + 2x + 2e^{4y} = k$.

How do we verify that $y$ implicitly defined by $x^2y^3 + 2x + 2e^{4y} = k$ is the solution of the differential equation

$$y' = -\frac{2xy^3 + 2}{3x^2y^2 + 8e^{4y}}?$$

We differentiate the whole equation $x^2y^3 + 2x + 2e^{4y} = k$ implicitly (CALCULUS I FOR ENGINEERS) and deduce the given differential equation.

**1.1.3 Integral Curves.**

A graph of a solution of a differential equation is called an **integral curve** of the equation. So if we know the general solution, then we obtain infinitely many integral curves, because of the arbitrary constant in the general solution.

**Example 1.1.5.** The general solution of $y' + y = 2$ is $y = 2 + ke^{-x}$ for all $x$. The figure 1.1 shows the integral curves with $k = -6, -3, 0, 3, 6$, i.e., the graphs of $y = 2 - 6e^{-x}$, $y = 2 - 3e^{-x}$, $y = 2$, $y = 2 + 3e^{-x}$ and $y = 2 + 6e^{-x}$.

**Exercise 1.1.6.** (1) Show that the differential equation $y' + y/x = e^x$ has the general solution

$$y = \frac{1}{x}(xe^x - e^x + e), \quad x \neq 0.$$

(2) Show that $y' + xy = 2$ has the general solution

$$y = e^{-x^2/2} \int_0^x 2e^{w^2/2} dw + ke^{-x^2/2}.$$

Use the **Matlab** or **Mathematica**, sketch the integral curves for various values of $k$. 

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1.1.4 Initial Value Problem.
Since a differential equation can have infinitely many integral curves, if we fix a point \((x_0, y_0)\) on the plane, then we may find only one integral curve passing through the fixed point \((x_0, y_0)\), which gives only one solution. From this observation, the problem solving a first-order differential equation

\[ F(x, y, y') = 0, \quad y(x_0) = y_0, \]

where \(x_0\) and \(y_0\) are given numbers, is called an initial value problem and the condition \(y(x_0) = y_0\) is called an initial condition. We need to keep in mind that the initial value problem has unique solution.

Example 1.1.7. Solve the initial value problem \(y' = e^{-x}\) and \(y(0) = 2\).

Answer. It is straightforward to see that the general solution of the equation \(y' = e^{-x}\) is \(y = -e^{-x} + k\), where \(k\) is any constant. Because of the initial condition, we have

\[ 2 = y(0) = -e^{-0} + k = -1 + k \quad \rightarrow \quad 2 = -1 + k \quad \rightarrow \quad k = 3. \]

Hence, the solution of the initial value problem is \(y = -e^{-x} + 3\). 

1.1.5 Direction Fields.
Skip. Please read the textbook.

\[ ^1 \text{the single one of its kind} \]
§1.2 Separable Equations.

**Definition 1.2.1.** A differential equation is called **separable** if it can be written

\[ y' = A(x)B(y). \]

How to solve a separable differential equation \( y' = A(x)B(y) \)?

**Theorem 1.2.2 (Strategy).** We separate the variables and integrate as follows:

\[
\frac{dy}{dx} = y' = A(x)B(y), \quad dy = A(x)B(y)dx, \quad \frac{1}{B(y)}dy = A(x)dx, \quad \int \frac{1}{B(y)}dy = \int A(x)dx.
\]

After integrating it, we simplify and get the general solution of the differential equation. **Be careful!** It may be or may not be possible to solve explicitly for \( y(x) \).

**Example 1.2.3.** Solve the differential equation \( y' = e^{-x}y^2 \).

**Answer.** Since the equation has the form of a separable equation, so it is a separable differential equation and we follow the strategy above: separate and integrate.

\[
\frac{dy}{dx} = y' = e^{-x}y^2, \quad dy = e^{-x}y^2dx, \quad \frac{1}{y^2}dy = e^{-x}dx \quad (y \neq 0)
\]

\[
\int \frac{1}{y^2}dy = \int e^{-x}dx, \quad -\frac{1}{y} = -e^{-x} + k, \quad \frac{1}{y} = e^{-x} - k, \quad y = \frac{1}{e^{-x} - k},
\]

where \( k \) is an arbitrary constant. Hence the general solution is

\[ y = \frac{1}{e^{-x} - k}. \]

**Remark 1.2.4.** In the solution above, we assumed \( y \neq 0 \) and found the general solution. We observe

1. For value of \( k \), the general solution cannot produce \( y = 0 \).
2. \( y = 0 \) is also a solution of the differential equation, because \( y(x) = 0 \) satisfies the equation \( y' = y^2e^{-x} \) for all \( x \).

These two observations defines a singular solution of a differential equation, i.e., the solution \( y(x) = 0 \) is called the **singular solution** of the equation, which cannot be deduced from the general solution.

Another example of the differential equation having a singular solution is

\[ y' = 6x(y - 1)^{2/3}. \]

Solve and check this out by yourself.

**Answer.** The general solution is \( y = 1 + (x^2 + C)^3 \) and the differential equation has a singular solution \( y(x) = 1 \).

**Example 1.2.5.** Solve the differential equation \( x^2y' = 1 + y \).

**Answer.** It is easy to see

\[ y' = \frac{1+y}{x^2} = \frac{1}{x^2} (1+y), \]

which is the form of a separable equation. So it is a separable equation. Using the strategy 1.2.2,

\[
\frac{dy}{dx} = y' = \frac{1}{x^2} (1+y), \quad dy = \frac{1}{x^2} (1+y)dx, \quad \int \frac{1}{1+y}dy = \int \frac{1}{x^2}dx \quad (1+y \neq 0)
\]
\[
\int \frac{1}{1+y} \, dy = \int \frac{1}{x^2} \, dx, \quad \ln |1+y| = -\frac{1}{x} + C, \quad 1 + y = De^{-\frac{1}{x}}, \quad y = -1 + De^{-\frac{1}{x}},
\]

where \( C \) and \( D = e^C \) are arbitrary constants. Hence the general solution is

\[
y = -1 + De^{-\frac{1}{x}}. \tag{1.2.1}
\]

Now we discuss the case when \( 1 + y = 0 \), i.e., \( y(x) = -1 \).

1. Since \( y(x) = -1 \) does satisfy the given differential equation, it is a solution of the equation.

2. If \( D = 0 \) in the general solution (1.2.1), then we have \( y = -1 \).

Because of the second observation, \( y = -1 \) cannot be a singular solution. It is just one particular solution. \( \square \)

**Example 1.2.6.** Solve the initial value problem \( y' = e^{-x}y^2 \) with \( y(3) = 4 \).

**Answer.** From the solution in the Example 1.2.3, the general solution is

\[
y = \frac{1}{e^{-x} - k},
\]

where \( k \) is an arbitrary constant. Using the given initial condition \( y(1) = 4 \), we determine the constant \( k \):

\[
4 = y(1) = \frac{1}{e^{-1} - k}, \quad 4 = \frac{e}{1 - ke}, \quad 1 - ke = \frac{e}{4}, \quad k = \frac{1 - e/4}{e} = \frac{1}{e} - \frac{1}{4}.
\]

Therefore, the solution of the initial value problem is

\[
y = \frac{1}{e^{-x} + 1/4 - 1/e}. \tag{1.2.2}
\]

**Example 1.2.7.** Solve the initial value problem

\[
y' = y \frac{(x - 1)^2}{y + 3}, \quad y(3) = -1.
\]

**Answer.** It is easy to see

\[
y' = y \frac{(x - 1)^2}{y + 3} = (x - 1)^2 \frac{y}{y + 3}
\]

which is the form of a separable equation. So it is a separable equation. Using the strategy 1.2.2,

\[
\frac{dy}{dx} = y' = (x - 1)^2 \frac{y}{y + 3}, \quad dy = (x - 1)^2 \frac{y}{y + 3} \, dx, \quad \frac{y + 3}{y} \, dy = (x - 1)^2 \, dx, \quad (y \neq 0)
\]

\[
\int \frac{y + 3}{y} \, dy = \int \left(1 + \frac{3}{y}\right) \, dy = \int (x - 1)^2 \, dx, \quad y + 3 \ln |y| = \frac{(x - 1)^3}{3} + k,
\]

where \( k \) is an arbitrary constant and the solution is implicitly defined. By the given initial condition,

\[
y(3) + 3 \ln |y(3)| = \frac{(3 - 1)^3}{3} + k, \quad -1 + 3 \ln |-1| = \frac{(3 - 1)^3}{3} + k, \quad -1 = \frac{8}{3} + k, \quad k = -\frac{11}{3}.
\]

Hence, the solution of the initial value problem is

\[
y + 3 \ln |y| = \frac{(x - 1)^3}{3} - \frac{11}{3}, \tag{1.2.2}
\]

which is implicitly defined. Since a logarithmic function is defined on positive real numbers, i.e., \( \ln |y| \) in the solution (1.2.2) is defined for \( y \neq 0 \), thus the condition \( y \neq 0 \) is satisfied. \( \square \)
Example 1.2.8. Solve the differential equations.

\( y' = 1 + x + y + xy. \)

\( y' = 6e^{2x-y} \) with \( y(0) = 0. \)

\( p(x)y' + q(x)y = q(x), \) where \( p(x) \neq 0 \) and \( q(x) \neq 0 \) are given functions.

\( p(x)y' + q(x)y^2 = q(x), \) where \( p(x) \neq 0 \) and \( q(x) \neq 0 \) are given functions.

\( e^{-3xy'} + x\sin(2y) = 0. \)

\( xy' + y = 1 \) (Final Exam of Fall 2009)

\( \Box \)

1.2.1 Extended Method: Reduction to Separable Form.

Certain nonseparable differential equations can be made separable by transformation that introduce for \( y \) a 
new unknown function, simply, a substitution. We discuss this technique for equations

\( y' = f \left( \frac{y}{x} \right), \quad (1.2.3) \)

where \( f \) is a differentiable function of \( y/x. \) This form of such a differential equation suggests that we set
\( u = y/x. \) Then the substitution implies, by the Product Rule,

\( y = ux, \quad \text{and} \quad y' = xu' + u. \)

Putting into the given equation (1.2.3), we get

\[ xu' + u = f(u), \quad \text{i.e.,} \quad xu' = f(u) - u, \quad \frac{du}{dx} = u' = \frac{f(u) - u}{x} = \frac{1}{x} (f(u) - u), \]

which is a separable equation.

Example 1.2.9. Solve the differential equation:

\( 2xyy' = y^2 - x^2, \)

which is not in the form of a separable equation.

**Answer.** When we divide the whole equation \( 2xy \), the given equation becomes

\[ y' = \frac{1}{2} \left( \frac{y}{x} - \frac{x}{y} \right), \quad (1.2.4) \]

Because of the form, we try the substitution \( u = y/x. \) Then we have \( y = ux \) and \( y' = xu' + u. \) Putting them
into the equation (1.2.4), we get

\[ xu' + u = \frac{1}{2} \left( u - \frac{1}{u} \right) = \frac{u^2 - 1}{2u}, \quad xu' = -\frac{u^2 + 1}{2u}, \]

which is a separable equation. Using the strategy 1.2.2 for a separable equation, we deduce

\[ \frac{dx}{u^2 + 1} = -\frac{1}{x} dx, \quad \int \frac{2u}{u^2 + 1} du = -\int \frac{1}{x} dx, \]

\[ \ln |u^2 + 1| = -\ln |x| + C, \quad \ln |u^2 + 1| = \ln \left| \frac{1}{x} \right| + C, \quad u^2 + 1 = \frac{D}{x}, \]

where \( C \) and \( D \neq 0 \) are arbitrary constants. Finally, putting \( u = y/x \) back to the result, we conclude

\[ \left( \frac{y}{x} \right)^2 + 1 = \frac{D}{x}, \quad y^2 + x^2 = Dx, \]
which gives the implicitly defined solution. A little bit more work (explicitly completing the square) shows that the equation $y^2 + x^2 = Dx$ is equivalent to

$$\left(x - \frac{D}{2}\right)^2 + y^2 = \left(\frac{D}{2}\right)^2,$$

of which graphs, i.e., integral curves of the differential equation, are clearly circles centered at $(D/2, 0)$ with radius $|D|/2$. 

**Example 1.2.10.** Solve the differential equations.

$\triangleright$ $xy' = x - y$.

$\triangleright$ $2xyy' = 3y^2 + x^2$ with $y(1) = 2$.

$\triangleright$ $xyy' = x^2 + 2y^2$ (Final Exam of Fall 2009)

□ 1.2.2 Some Applications of Separable Differential Equations.

Skip. Please read the textbook.
§1.3 Linear Differential Equations.

**Definition 1.3.1.** A first–order differential equation is called to be **linear** if it has the form

\[ y' + p(x)y = q(x), \]

where \( p(x) \) and \( q(x) \) are assumed to be continuous.

How to solve a first–order linear differential equation \( y' + p(x)y = q(x) \)?

**Theorem 1.3.2 (Strategy: Integrating Factor \( e^{\int p(x)dx} \)).** Multiplying the differential equation by \( e^{\int p(x)dx} \), we get

\[ e^{\int p(x)dx}y' + e^{\int p(x)dx}p(x)y = q(x)e^{\int p(x)dx}. \]

Then the left–hand side of the equation becomes the derivative of the product \( e^{\int p(x)dx}y \). That is, the equation becomes

\[ \left(e^{\int p(x)dx}y\right)' = q(x)e^{\int p(x)dx}. \]

After integrating both sides, we obtain

\[ e^{\int p(x)dx}y = \int \left(q(x)e^{\int p(x)dx}\right) dx + C, \]

where \( C \) is the constant of integration. Therefore, we deduce the general solution

\[ y = e^{-\int p(x)dx} \int \left(q(x)e^{\int p(x)dx}\right) dx + Ce^{-\int p(x)dx}. \quad (1.3.1) \]

**Remark 1.3.3.**

1. The function \( e^{\int p(x)dx} \) is called the **integrating factor** of the differential equation. It is not recommended memorizing the formula 1.3.1. However, it is suggested that you should remember the integrating factor \( e^{\int p(x)dx} \) and the steps to deduce the general solution.

2. When can a separable differential equation \( y' = A(x)B(y) \) be linear? When can a linear differential equation \( y' + p(x)y = q(x) \) be separable? If a differential equation is both separable and linear, then which strategy should we use to solve the differential equation?

In the linear differential equation, \( y' + p(x)y = q(x) \), if \( q(x) = 0 \), then the differential equation becomes separable. The strategy 1.3.2 or the strategy 1.2.2 on a separable differential equation implies the general solution

\[ y = Ce^{-\int p(x)dx}. \]

**Example 1.3.4.** Solve the differential equation \( y' + y = \sin x \).

**Answer.** Since the equation has the form of a linear equation with \( p(x) = 1 \) and \( q(x) = \sin x \), so it is a linear differential equation and we follow the strategy 1.3.2, i.e., integrating factor

\[ e^{\int p(x)dx} = e^{\int 1dx} = e^x. \]

Multiplying the equation by the integrating factor \( e^x \), we have

\[ e^x y' + e^x y = e^x \sin x, \quad i.e., \quad (e^x y)' = e^x \sin x. \]

Integrating both sides implies

\[ e^x y = \int e^x \sin x dx = \frac{1}{2} e^x (\sin x - \cos x) + C, \]

by the **Integration by Parts** formula in **Calculus I** and \( C \) is the constant of integration. Finally, we simplify and get the general solution

\[ y = \frac{1}{2} (\sin x - \cos x) + Ce^{-x}. \]
Example 1.3.5. Solve the initial value problem
\[ y' = 3x^2 - \frac{y}{x}, \quad y(1) = 5. \]

**Answer.** We observe the equation is same as
\[ y' + \frac{1}{x}y = 3x^2. \] (1.3.2)

Since the equation has the form of a linear equation with \( p(x) = 1/x \) and \( q(x) = 3x^2 \), so it is a linear differential equation and we follow the strategy 1.3.2, i.e., integrating factor
\[ e^{\int p(x) \, dx} = e^{\int 1/x \, dx} = e^{\ln x} = x, \quad \text{for} \quad x > 0. \]

Multiplying the equation (1.3.2) by the integrating factor \( x \), we have
\[ xy' + y = 3x^3, \quad \text{i.e.,} \quad (xy)' = 3x^3. \]

Integrating both sides implies
\[ xy = \int 3x^3 \, dx = \frac{3}{4}x^4 + C, \quad \text{i.e.,} \quad y = \frac{3}{4}x^3 + \frac{C}{x}. \]

By the initial condition \( y(1) = 5 \), we have
\[ 5 = y(1) = \frac{3}{4} \cdot 1^3 + \frac{C}{1}, \quad C = \frac{17}{4}. \]

Finally, the general solution is, for \( x > 0 \),
\[ y = \frac{3}{4}x^3 + \frac{17}{4x}. \]

In the solution of this example, we have chosen \( x > 0 \) in computing the integrating factor. Why did we choose \( x > 0 \) rather than \( x < 0 \)? The reason lies on the initial condition \( y(1) = 5 \). As you may recall from **SECTION 1.1 PRELIMINARY CONCEPTS**, the initial condition restricts our concerns to certain integral curve passing through the point \((x, y) = (1, 5)\). In other words, since we want to find the solution whose graph passes through the point \((x, y) = (1, 5)\), we have to choose \( x > 0 \). Suppose the initial condition \( y(-4) = 10 \) is given. Then the integral factor becomes \(-x\) and the general solution is, for \( x < 0 \),
\[ y = \frac{3}{4}x^3 - \frac{232}{x}. \]

Can we always have the general solution in which all the integrals can be evaluated? The answer is NO. For instance, the linear differential equation,
\[ y' + xy = 2, \]
has the integrating factor
\[ e^{\int x \, dx} = e^{\frac{x^2}{2}} \]
and the general solution is
\[ y = 2e^{\frac{x^2}{2}} \int e^{\frac{x^2}{2}} \, dx + Ce^{-\frac{x^2}{2}} \]
where \( \int e^{\frac{x^2}{2}} \, dx \) cannot be evaluated explicitly in elementary terms.

**Example 1.3.6.** Solve the differential equations.
- \( y' - xy = e^{2x} \).
- \( y' + y \tan x = \sin(2x) \) with \( y(0) = 1 \).
- \( x^2y' + xy = \sin x \) with \( y(1) = y_0 \), where \( y_0 \) is a given real number.
- \( xy' + y = 3xy \) with \( y(1) = 0 \).
- \( y' = (1 - y) \cos x \) with \( y(\pi) = 2 \).
- \( y' = 1 + x + y + xy \) with \( y(0) = 0 \).
- \( xy' - 2y = x^2 \). (Final Exam of Fall 2009)
- \( 2xy' - 3y = 9x^3 \) with \( y(1) = 0 \). (Final Exam of Fall 2009)
§1.4 Exact Differential Equations.
Compared to previous three sections, this section is difficult and long. So please focus and try to understand all examples. We will have the implicitly defined solution. The chain rule for partial derivatives in Calculus II will be used.

We have seen that a general solution \( y(x) \) of a first–order differential equation is often defined implicitly by an equation of the form

\[
\phi(x, y(x)) = C, \tag{1.4.1}
\]

where \( C \) is a constant. On the other hand, given the equation (1.4.1), we can recover the original differential equation by differentiating each side with respect to \( x \). Provided that the equation (1.4.1) implicitly defines \( y \) as a differentiable function of \( x \), this gives the original differential equation in the form

\[
\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y} \frac{dy}{dx} = \frac{\partial}{\partial x} C = 0, \quad M(x, y) + N(x, y) \frac{dy}{dx} = 0, \quad M(x, y)dx + N(x, y)dy = 0, \tag{1.4.2}
\]

where

\[
M(x, y) = \frac{\partial \phi(x, y)}{\partial x} = \phi_x(x, y) \quad \text{and} \quad N(x, y) = \frac{\partial \phi(x, y)}{\partial y} = \phi_y(x, y).
\]

The last equation in (1.4.2) is called the differential form. The general first–order differential equation \( y' = f(x, y) \) can be written in this form with \( M = f(x, y) \) and \( N = -1 \) or \( M = -f(x, y) \) and \( N = 1 \). The preceding discussion shows that, if there exists a function \( \phi(x, y) \) such that

\[
\frac{\partial \phi}{\partial x} = M \quad \text{and} \quad \frac{\partial \phi}{\partial y} = N,
\]

then the equation

\[
\phi(x, y) = C
\]

implicitly defines a general solution of the equation (1.4.2). In this case, the equation (1.4.2) is called an exact differential equation.

With this background, we make the following definitions.

**Definition 1.4.1.** A function \( \phi(x, y) \) is called a potential function for the differential equation

\[
M(x, y) + N(x, y)y' = 0 \quad \text{or} \quad M(x, y)dx + N(x, y)dy = 0
\]

on a region \( R \) of the plane if, for each \( (x, y) \) in \( R \),

\[
\frac{\partial \phi}{\partial x} = M \quad \text{and} \quad \frac{\partial \phi}{\partial y} = N(x, y).
\]

**Definition 1.4.2.** When a potential function exists on a region \( R \) for the differential equation

\[
M(x, y) + N(x, y)y' = 0,
\]

this equation is said to be exact on \( R \).

**Remark 1.4.3 (QUESTIONS).**

1. How can we determine whether the differential equation (1.4.2) is exact? That is, how do we know the existence of a potential function? The answer is given in the theorem 1.4.4 below.
2. If a differential equation is exact, i.e., if the potential function exists, then how can we find it? That is, how can we find the function \( \phi(x, y) \) such that

\[
\frac{\partial \phi}{\partial x} = M(x, y) \quad \text{and} \quad \frac{\partial \phi}{\partial y} = N(x, y)?
\]

The answer will be explained through the examples.
**Theorem 1.4.4 (Criterion/Test for Exactness).** Suppose that the functions \( M(x,y) \) and \( N(x,y) \) are continuous and have continuous first–order partial derivatives on a region \( R \) of the plane. Then the differential equation

\[
M(x,y) + N(x,y)y' = 0
\]

is exact in \( R \) if and only if

\[
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
\]

(1.4.3)

at each point of \( R \). That is, there exists a potential function \( \varphi(x,y) \) defined on \( R \) with \( \varphi_x = M \) and \( \varphi_y = N \) if and only if the equation (1.4.3) holds on \( R \).

**Example 1.4.5.** Solve the differential equation

\[
6xy - y^3 + (4y + 3x^2 - 3xy^2)y' = 0 \quad \text{or} \quad (6xy - y^3) \, dx + (4y + 3x^2 - 3xy^2) \, dy = 0,
\]

which is neither separable nor linear.

**Answer.** **Step 1. Test for Exactness.** Let \( M(x,y) = 6xy - y^3 \) and \( N(x,y) = 4y + 3x^2 - 3xy^2 \). Then

\[
\frac{\partial M}{\partial y} = 6x - 3y^2 = \frac{\partial N}{\partial x}.
\]

So by the Theorem 1.4.4, the given equation is exact.

**Step 2. Solution.** Again by the Theorem 1.4.4, a potential function \( \varphi \) satisfies \( \varphi_x = M \) and \( \varphi_y = N \), i.e.,

\[
\frac{\partial \varphi}{\partial x} = 6xy - y^3, \quad \text{and} \quad \frac{\partial \varphi}{\partial y} = 4y + 3x^2 - 3xy^2.
\]

(1.4.4)

To find \( \varphi \), we integrate either one. Let us integrate the first one \( \varphi_x \) with respect to \( x \):

\[
\varphi(x,y) = \int \left( \frac{\partial \varphi}{\partial x} \right) \, dx = \int (6xy - y^3) \, dx = 3x^2y - xy^3 + g(y),
\]

thinking of the function \( g(y) \) as an “arbitrary constant of integration”, as far as the variable \( x \) is concerned. Inserting this result to the second equation in (1.4.4), we determine the function \( g(y) \):

\[
4y + 3x^2 - 3xy^2 = \frac{\partial \varphi}{\partial y} = \frac{\partial}{\partial y} (3x^2y - xy^3 + g(y)) = 3x^2 - 3xy^2 + g'(y),
\]

\[
4y + 3x^2 - 3xy^2 = 3x^2 - 3xy^2 + g'(y), \quad g'(y) = 4y, \quad g(y) = \int 4y \, dy = 2y^2 + C.
\]

Thus, finally we obtain the potential function \( \varphi(x,y) \):

\[
\varphi(x,y) = 3x^2y - xy^3 + 2y^2 + C
\]

and the general solution of the differential equation

\[
\varphi(x,y) = 3x^2y - xy^3 + 2y^2 + C = D, \quad \text{i.e.,} \quad 3x^2y - xy^3 + 2y^2 = E,
\]

where \( C, D \) and \( E = D - C \) are arbitrary constants.

**Step 3. Checking.** We check this implicitly defined solution by implicitly differentiating and see whether it leads to the given differential equation:

\[
0 = \frac{d}{dx} E = \frac{d}{dx} (3x^2y - xy^3 + 2y^2) = 6xy + 3x^2y' - y^3 - 3xy^2y' + 4yy' = 6xy - y^3 + (4y + 3x^2 - 3xy^2) \, y'.
\]

This is the given differential equation and so the implicitly defined function \( y \) in Step 2 above is really the general solution. \[\square\]
In Step 2 of the solution above,

1. we have integrated \( \varphi_x \) with respect to \( x \) and
2. found \( \varphi \) having \( g(y) \) and
3. by differentiating \( \varphi \) with respect to \( y \), we found \( g(y) \) and finally \( \varphi \).

However, we can argue in the other way, i.e.,

1. integrate \( \varphi_y \) with respect to \( y \) and
2. find \( \varphi \) having \( h(x) \) and
3. by differentiating \( \varphi \) with respect to \( x \), we can find \( h(x) \) and finally \( \varphi \).

For the argument in detail, see the example below.

**Example 1.4.6.** Solve the differential equation

\[
\cos(x + y) + (3y^2 + 2y + \cos(x + y))y' = 0 \quad \text{or} \quad \cos(x + y)dx + (3y^2 + 2y + \cos(x + y)) \, dy = 0.
\]

**Answer.** **Step 1. Test for Exactness.** Let \( M(x, y) = \cos(x + y) \) and \( N(x, y) = 3y^2 + 2y + \cos(x + y) \).

\[
\frac{\partial M}{\partial y} = -\sin(x + y) = \frac{\partial N}{\partial x}.
\]

So by the Theorem 1.4.4, the given equation is exact.

**Step 2. Solution.** Again by the Theorem 1.4.4, a potential function \( \varphi \) satisfies \( \varphi_x = M \) and \( \varphi_y = N \), i.e.,

\[
\frac{\partial \varphi}{\partial x} = \cos(x + y), \quad \text{and} \quad \frac{\partial \varphi}{\partial y} = 3y^2 + 2y + \cos(x + y).
\]

(1.4.5)

To find \( \varphi \), let us integrate the second one \( \varphi_y \) with respect to \( y \):

\[
\varphi(x, y) = \int \left( \frac{\partial \varphi}{\partial y} \right) \, dy = \int (3y^2 + 2y + \cos(x + y)) \, dy = y^3 + y^2 + \sin(x + y) + h(x),
\]

thinking of the function \( h(x) \) as an “arbitrary constant of integration”, as far as the variable \( y \) is concerned. Inserting this result to the first equation in (1.4.5), we determine the function \( h(x) \):

\[
\cos(x + y) = \frac{\partial \varphi}{\partial x} = \frac{\partial}{\partial x} \left( y^3 + y^2 + \sin(x + y) + h(x) \right) = \cos(x + y) + h'(x),
\]

\[
\cos(x + y) = \cos(x + y) + h'(x), \quad h'(x) = 0, \quad h(x) = C.
\]

Thus, finally we obtain the potential function \( \varphi(x, y) \):

\[
\varphi(x, y) = y^3 + y^2 + \sin(x + y) + C
\]

and the general solution of the differential equation

\[
\varphi(x, y) = y^3 + y^2 + \sin(x + y) + C = D, \quad \text{i.e.,} \quad y^3 + y^2 + \sin(x + y) = E,
\]

where \( C, D \) and \( E = D - C \) are arbitrary constants.

**Step 3. Checking.** We check this implicitly defined solution by implicitly differentiating and see whether it leads to the given differential equation:

\[
0 = \frac{d}{dx} E = \frac{d}{dx} (y^3 + y^2 + \sin(x + y))
\]

\[
= 3y^2 y' + 2yy' + \cos(x + y) + \cos(x + y) y' = \cos(x + y) + (3y^2 + 2y + \cos(x + y)) y'.
\]

This is the given differential equation and so the implicitly defined function \( y \) in Step 2 above is really the general solution.
Example 1.4.7. Solve the differential equation
\[ 2xy^3 + 2 + (3x^2y^2 + 8e^{4y})y' = 0 \quad \text{or} \quad (2xy^3 + 2) \, dx + (3x^2y^2 + 8e^{4y}) \, dy = 0. \]

**ANSWER. Step 1. Test for Exactness.** Let \( M(x, y) = 2xy^3 + 2 \) and \( N(x, y) = 3x^2y^2 + 8e^{4y} \). Then
\[
\frac{\partial M}{\partial y} = 6xy^2 = \frac{\partial N}{\partial x}.
\]

So by the Theorem 1.4.4, the given equation is exact.

**Step 2. Solution.** Again by the Theorem 1.4.4, a potential function \( \varphi \) satisfies \( \varphi_x = M \) and \( \varphi_y = N \), i.e.,
\[
\frac{\partial \varphi}{\partial x} = 2xy^3 + 2, \quad \text{and} \quad \frac{\partial \varphi}{\partial y} = 3x^2y^2 + 8e^{4y}. \quad (1.4.6)
\]

To find \( \varphi \), we integrate either one. Let us integrate the first one \( \varphi_x \) with respect to \( x \):
\[
\varphi(x, y) = \int \left( \frac{\partial \varphi}{\partial x} \right) \, dx = \int (2xy^3 + 2) \, dx = x^2y^3 + 2x + g(y),
\]

thinking of the function \( g(y) \) as an “arbitrary constant of integration”, as far as the variable \( x \) is concerned. Inserting this result to the second equation in (1.4.6), we determine the function \( g(y) \):
\[
3x^2y^2 + 8e^{4y} = \frac{\partial \varphi}{\partial y} = \frac{\partial}{\partial y} (x^2y^3 + 2x + g(y)) = 3x^2y^2 + g'(y),
\]
\[
3x^2y^2 + 8e^{4y} = 3x^2y^2 + g'(y), \quad g'(y) = 8e^{4y}, \quad g(y) = \int 8e^{4y} \, dy = 2e^{4y} + C.
\]

Thus, finally we obtain the potential function \( \varphi(x, y) \):
\[
\varphi(x, y) = x^2y^3 + 2x + 2e^{4y} + C
\]

and the general solution of the differential equation
\[
\varphi(x, y) = x^2y^3 + 2x + 2e^{4y} + C = D, \quad \text{i.e.,} \quad x^2y^3 + 2x + 2e^{4y} = E,
\]

where \( C, D \) and \( E = D - C \) are arbitrary constants.

**Step 3. Checking.** We check this implicitly defined solution by implicitly differentiating and see whether it leads to the given differential equation:
\[
0 = \frac{d}{dx} E = \frac{d}{dx} (x^2y^3 + 2x + 2e^{4y})
\]
\[
= 2xy^3 + 3x^2y^2y' + 2 + 8e^{4y}y' = 2xy^3 + 2 + (3x^2y^2 + 8e^{4y}) \, y'.
\]

This is the given differential equation and so the implicitly defined function \( y \) in Step 2 above is really the general solution. \( \square \)

Example 1.4.8. Solve the differential equation
\[ x^2 + 3xy + (4xy + 2x)y' = 0 \quad \text{or} \quad (x^2 + 3xy) \, dx + (4xy + 2x) \, dy = 0. \]

**ANSWER. Step 1. Test for Exactness.** Let \( M(x, y) = x^2 + 3xy \) and \( N(x, y) = 4xy + 2x \). Then
\[
\frac{\partial M}{\partial y} = 3x, \quad \text{and} \quad \frac{\partial N}{\partial x} = 4y + 2.
\]

We have the equality \( M_y = N_x \) along the straight line \( 3x = 4y + 2 \), but the equality does not hold for every point in a region of the plane. Hence the differential equation is not exact and we cannot solve the differential equation using the strategy above. \( \square \)
Example 1.4.9. Solve the differential equation

\[ e^x \sin y - 2x + (e^x \cos y + 1) y' = 0 \quad \text{or} \quad (e^x \sin y - 2x) \, dx + (e^x \cos y + 1) \, dy = 0, \]

which is neither separable nor linear.

**Answer.** Step 1. Test for Exactness. Let \( M(x,y) = e^x \sin y - 2x \) and \( N(x,y) = e^x \cos y + 1 \). Then

\[
\frac{\partial M}{\partial y} = e^x \cos y = \frac{\partial N}{\partial x}.
\]

So by the Theorem 1.4.4, the given equation is exact.

Step 2. Solution. Again by the Theorem 1.4.4, a potential function \( \phi \) satisfies \( \phi_x = M \) and \( \phi_y = N \), i.e.,

\[
\frac{\partial \phi}{\partial x} = e^x \sin y - 2x, \quad \text{and} \quad \frac{\partial \phi}{\partial y} = e^x \cos y + 1. \tag{1.4.7}
\]

To find \( \phi \), we integrate either one. Let us integrate the first one \( \phi_x \) with respect to \( x \):

\[ \phi(x,y) = \int \left( \frac{\partial \phi}{\partial x} \right) \, dx = \int (e^x \sin y - 2x) \, dx = e^x \sin y - x^2 + g(y), \]

thinking of the function \( g(y) \) as an “arbitrary constant of integration”, as far as the variable \( x \) is concerned. Inserting this result to the second equation in (1.4.7), we determine the function \( g(y) \):

\[ e^x \cos y + 1 = \frac{\partial \phi}{\partial y} = \frac{\partial}{\partial y} \left( e^x \sin y - x^2 + g(y) \right) = e^x \cos y + g'(y), \]

\[ e^x \cos y + 1 = e^x \cos y + g'(y), \quad g'(y) = 1, \quad g(y) = \int 1 \, dy = y + C. \]

Thus, finally we obtain the potential function \( \phi(x,y) \):

\[ \phi(x,y) = e^x \sin y - x^2 + y + C \]

and the general solution of the differential equation

\[ \phi(x,y) = e^x \sin y - x^2 + y + C = D, \quad \text{i.e.,} \quad e^x \sin y - x^2 + y = E, \]

where \( C, D \) and \( E = D - C \) are arbitrary constants.

Step 3. Checking. We check this implicitly defined solution by implicitly differentiating and see whether it leads to the given differential equation:

\[ 0 = \frac{d}{dx} E = \frac{d}{dx} \left( e^x \sin y - x^2 + y \right) = e^x \sin y + e^x (\cos y) y' - 2x + y' = e^x \sin y - 2x + (e^x \cos y + 1) y'. \]

This is the given differential equation and so the implicitly defined function \( y \) in Step 2 above is really the general solution.

Before we end the section, let us bring the caution through an example.

Example 1.4.10. Solve the differential equation

\[ y^3 + 3xy^2 y' = 0 \quad \text{or} \quad y^3 \, dx + 3xy^2 \, dy = 0. \]
ANSWER 1. SOLUTION OF EXACT EQUATION.

Step 1. Test for Exactness. Let \( M(x,y) = y^3 \) and \( N(x,y) = 3xy^2 \). Then
\[
\frac{\partial M}{\partial y} = 3y^2 = \frac{\partial N}{\partial x}.
\]
So by the Theorem 1.4.4, the given equation is exact.

Step 2. Solution. Again by the Theorem 1.4.4, a potential function \( \phi \) satisfies
\[
\frac{\partial \phi}{\partial x} = M \quad \text{and} \quad \frac{\partial \phi}{\partial y} = N,
\]
\[\text{i.e.,} \]
\[
\frac{\partial \phi}{\partial x} = y^3, \quad \text{and} \quad \frac{\partial \phi}{\partial y} = 3xy^2. \tag{1.4.8}
\]
To find \( \phi \), we integrate either one. Let us integrate the first one \( \phi_x \) with respect to \( x \):
\[
\phi(x,y) = \int \left( \frac{\partial \phi}{\partial x} \right) \, dx = \int y^3 \, dx = xy^3 + g(y).
\]
Inserting this result to the second equation in (1.4.8), we determine the function \( g(y) \):
\[
3xy^2 = \frac{\partial \phi}{\partial y} = \frac{\partial}{\partial y} (xy^3 + g(y)) = 3xy^2 + g'(y),
\]
\[
3xy^2 = 3xy^2 + g'(y), \quad g'(y) = 0, \quad g(y) = C.
\]
Thus, finally we obtain the potential function \( \phi(x,y) \):
\[
\phi(x,y) = xy^3 + C
\]
and the general solution of the differential equation
\[
\phi(x,y) = xy^3 + C = D, \quad \text{i.e.,} \quad xy^3 = E, \tag{1.4.9}
\]
where \( C, D \) and \( E = D - C \) are arbitrary constants.

Step 3. Checking. We check this implicitly defined solution by implicitly differentiating and see whether it leads to the given differential equation:
\[
0 = \frac{d}{dx} E = \frac{d}{dx} (xy^3) = y^3 + 3xy^2 y'.
\]
This is the given differential equation and so the implicitly defined function \( y \) in Step 2 above is really the general solution. \( \square \)

Suppose that we divide each term of the differential equation in the example above by \( y^2 \) to obtain
\[
y + 3xy' = 0 \quad \text{or} \quad y dx + 3x dy = 0.
\]
Then this equation is not exact, because, with \( M(x,y) = y \) and \( N(x,y) = 3x \), we have
\[
\frac{\partial M}{\partial y} = 1 \neq 3 = \frac{\partial N}{\partial x}.
\]
That is, it does not pass the Test for Exactness 1.4.3. So it is suggested that we should not modify the given differential equation by dividing by common factors. However, in the example above, due to the common factors, we can solve the differential equation as follows.
Answer 2. Solution of Separable Equation. We observe

\[ y^3 + 3xy^2y' = 0, \quad y' = -\frac{y^3}{3xy^2} = -\frac{1}{3x}y, \quad (x \neq 0) \]

which means the given equation is separable. So we use the strategy for the separable equation.

\[ \frac{dy}{dx} = -\frac{1}{3x}y, \quad \frac{1}{y} dy = -\frac{1}{3x} dx, \quad \int \frac{1}{y} dy = -\int \frac{1}{3x} dx, \quad (y \neq 0) \]

\[ \ln |y| = -\frac{1}{3} \ln |x| + C, \quad |y| = D|x|^{-1/3}, \quad xy^3 = E, \]

where \( D = e^C \) and \( E = D^3 \) are arbitrary constants and we assumed \( x > 0 \) and \( y > 0 \). As we can see, the solution is exactly same as the one (1.4.9) in Answer 1. Solution of Exact Equation. \( \square \)

Remark 1.4.11 (Important Caution). If \( \varphi \) is a potential function for \( M + Ny' = 0 \), then \( \varphi \) itself is not the solution. The general solution is defined implicitly by the equation \( \varphi(x, y) = C \).

Exercise 1.4.12. Solve the differential equations.

\( \triangleright \) \( 2x + 3y + (3x + 2y)y' = 0. \) \hspace{1cm} Answer. \( x^2 + 3xy + y^2 = C. \)

\( \triangleright \) \( (1 + ye^y)dx + (2y + xe^y)dy = 0. \) \hspace{1cm} Answer. \( x + e^y + y^2 = C. \)

\( \triangleright \) \( 3x^2 + 2y^2 + (4xy + 6y^2)y' = 0. \) \hspace{1cm} Answer. \( x^3 + 2xy^2 + 2y^3 = C. \)

\( \triangleright \) \( (\cos x + \ln y)dx + \left(\frac{x}{y} + e^y\right)dy = 0. \) (Final of Fall 2009) \hspace{1cm} Answer. \( \sin x + x\ln y + e^y = C. \)

\( \square \) 1.4.1 Appendix: Formula.

For an exact differential equation \( Mdx + Ndy = 0 \), we discuss an easier(?) way to find the potential function \( \varphi(x, y) \) and so the implicitly defined solution. Personally, I do not recommend the formula(?) below.

Let \( \mathcal{M} = \int M \, dx \) and \( \mathcal{N} = \int N \, dy \). Since the potential function \( \varphi(x, y) \) satisfies \( \varphi_x = M \), we just integrate \( M \) with respect to \( x \) and add \( g(y) \),

\[ \varphi(x, y) = \int M \, dx + g(y) = \mathcal{M} + g(y). \quad (1.4.10) \]

Because of the relations \( \varphi_x = M \) and \( \varphi_y = N \), to find \( g(y) \), we differentiate the equation (1.4.10) with respect to \( y \) and compare with \( N \),

\[ \mathcal{M}_y + g'(y) = \frac{\partial}{\partial y} (\mathcal{M} + g(y)) = N, \quad g'(y) = N - \mathcal{M}_y, \]

\[ g(y) = \int (N - \mathcal{M}_y) \, dy = \int N \, dy - \int \mathcal{M}_y \, dy = \mathcal{N} - \int \mathcal{M}_y \, dy. \]

Therefore, we deduce a formula (if one wants to call it so)

\[ \varphi = \mathcal{M} + \mathcal{N} - \int \mathcal{M}_y \, dy \]

and the implicitly defined solution is

\[ \mathcal{M} + \mathcal{N} - \int \mathcal{M}_y \, dy = C. \]

Example 1.4.13. Solve the exact differential equation

\[ (\cos x + \ln y)dx + \left(\frac{x}{y} + e^y\right)dy = 0. \]
**Answer.** Since the problem says that the given equation is exact. So we do not have to test for exactness. We just find the potential function and the implicitly defined solution. Let us use the formula above.

Since \( M = \cos x + \ln y \) and \( N = x / y + e^y \), so

\[
\mathcal{M} = \int M \, dx = \int (\cos x + \ln y) \, dx = \sin x + x \ln y, \\
\mathcal{M}_y = \frac{x}{y}, \quad \int \mathcal{M}_y \, dy = \int \frac{x}{y} \, dy = x \ln y, \\
\mathcal{N} = \int N \, dy = \int \left( \frac{x}{y} + e^y \right) \, dy = x \ln y + e^y.
\]

(We do not have to add the constants of integration.) The formula above implies the potential function \( \varphi \) and the implicitly defined solution,

\[
\varphi = \sin x + x \ln y + x \ln y + e^y - x \ln y = \sin x + x \ln y + e^y, \quad \sin x + x \ln y + e^y = C.
\]

□
§1.5 Integrating Factors.

Let us start with the differential equation

\[ -y + xy' = 0, \]  

(1.5.1)

which is not an exact differential equation. However, if we multiply the whole equation by \( \frac{1}{x^2} \), then the equation becomes

\[ -\frac{y}{x^2} + \frac{1}{x} y' = 0 \]

and we observe the left–hand side of the equation is, in fact, the derivative of \( y/x \), i.e.,

\[ \frac{d}{dx} \left( \frac{y}{x} \right) = -\frac{y}{x^2} + \frac{1}{x} y' = 0, \quad i.e., \quad \frac{y}{x} = C. \]

That is, the given differential equation (1.5.1) has the general solution \( y/x = C \). Check: Implicitly differentiating \( y/x = C \), we get

\[ -\frac{y}{x^2} + \frac{1}{x} y' = 0, \quad -y + xy' = 0. \]

Here is another example.

**Example 1.5.1.** Solve the differential equation

\[ y^2 - 6xy + (3xy - 6x^2)y' = 0, \]  

(1.5.2)

which is neither separable, linear, nor exact.

**Answer.** **Step 1. Multiply by** \( \mu(x,y) = y \). When we multiply the equation by \( \mu(x,y) = y \), the given equation becomes

\[ y(y^2 - 6xy) + y(3xy - 6x^2)y' = 0, \quad i.e., \quad y^3 - 6xy^2 + (3xy^2 - 6x^2)y' = 0. \]  

(1.5.3)

We observe the resulting equation is exact, because with \( M(x,y) = y^3 - 6xy^2 \) and \( N(x,y) = 3xy^2 - 6x^2y \),

\[ \frac{\partial M}{\partial y} = 3y^2 - 12xy = \frac{\partial N}{\partial x}. \]

**Step 2. Solution.** By the strategy for the exact differential equation, we deduce the potential function \( \phi(x,y) \) and the implicitly defined solution of the equation (1.5.3) as follows:

\[ \phi(x,y) = \int M \, dx = \int (y^3 - 6xy^2) \, dx = xy^3 - 3x^2y^2 + g(y). \]

The partial derivative of \( \phi(x,y) \) with respect to \( y \) should be \( N \), i.e.,

\[ 3xy^2 - 6x^2y = N = \frac{\partial \phi(x,y)}{\partial y} = \frac{\partial}{\partial y} (xy^3 - 3x^2y^2 + g(y)) = 3xy^2 - 6x^2y + g'(y), \]

\[ i.e., \quad g'(y) = 0, \quad g(y) = C. \]

Hence, we deduce the potential function

\[ \phi(x,y) = xy^3 - 3x^2y^2 + C \]

and the general solution defined implicitly,

\[ \phi(x,y) = D, \quad xy^3 - 3x^2y^2 + C = D, \quad xy^3 - 3x^2y^2 = E. \]

Wherever \( y \neq 0 \), this defines the general solution of the original non–exact equation (1.5.2) given in the problem.
Let us review the solution above.

1. The given differential equation was not exact.
2. Multiplying the whole equation by \( m(x, y) \), we made the given equation to be exact.
3. By the strategy for an exact equation, we found the general solution of the modified equation (1.5.3), which was eventually same as the solution of the original equation (1.5.2).

The function \( m(x, y) \) has played a very important role in the solution above and so we should give a name to the function \( m(x, y) \).

**Definition 1.5.2.** Let \( M(x, y) \) and \( N(x, y) \) be defined on a region \( R \) of the plane. Then \( m(x, y) \) is an **integrating factor** for \( M(x, y) + N(x, y)y' = 0 \) if

1. \( m(x, y) \neq 0 \) for all \( (x, y) \in R \) and
2. \( mM + mNy' = 0 \) is exact on \( R \).

How to find an integrating factor of the differential equation \( M + Ny' = 0 \)?

**Theorem 1.5.3 (Strategy).** In order for a function \( m(x, y) \) to be an integrating factor, it should make the equation 

\[
M + Ny' = 0
\]

to be exact (in some region of the plane), which means \( mM \) and \( mN \) should satisfy the **Test for Exactness 1.4.4**, i.e.,

\[
\frac{\partial}{\partial y} (mM) = \frac{\partial}{\partial x} (mN)
\]

in the region. Thus, we solve the differential equation (1.5.4) to find the integrating factor \( m \).

**Example 1.5.4.** Solve the differential equation

\[
x - xy - y' = 0.
\]

**Answer. Step 1. Test for Exactness.** Let \( M(x, y) = x - xy \) and \( N(x, y) = -1 \). Then

\[
\frac{\partial M}{\partial y} = -x, \quad \text{but} \quad \frac{\partial N}{\partial x} = 0.
\]

So the given equation is not exact.

**Step 2. Integrating Factor** \( \mu(x, y) \). We solve the following equation for \( \mu(x, y) \):

\[
\frac{\partial}{\partial y} (\mu M) = \frac{\partial}{\partial x} (\mu N), \quad \frac{\partial}{\partial y} (\mu (x - xy)) = \frac{\partial}{\partial x} (-\mu), \quad (x - xy) \frac{\partial \mu}{\partial y} - x \mu = -\frac{\partial \mu}{\partial x}.
\]

We choose \( \mu(x, y) \) such that \( \mu_y = 0 \). Then \( \mu(x, y) \) becomes a function of only \( x \), i.e., \( \mu(x, y) = \mu(x) \), and so the equation becomes simple to be solved

\[
-x \mu, \quad x \mu,
\]

which is a separable equation with the solution \( \mu(x, y) = Ce^{x^2/2} \). Since we need just one integrating factor, we choose \( C = 1 \), i.e., \( \mu(x, y) = e^{x^2/2} \).

**Step 3. Solving Exact Equation.** Multiplying the original equation (1.5.5) by \( \mu(x, y) = e^{x^2/2} \),

\[
(x - xy)e^{x^2/2} - e^{x^2/2}y' = 0,
\]

(1.5.6)
which is an exact differential equation. A potential function \( \varphi \) satisfies
\[
\frac{\partial \varphi}{\partial x} = (x - xy)e^{x^2/2}, \quad \text{and} \quad \frac{\partial \varphi}{\partial y} = -e^{x^2/2}.
\] (1.5.7)

To find \( \varphi \), we integrate either one. Let us integrate the second one \( \varphi_y \) with respect to \( y \):
\[
\varphi(x, y) = \int \left( \frac{\partial \varphi}{\partial y} \right) dy = \int \left( -e^{x^2/2} \right) dy = -e^{x^2/2}y + h(x).
\]

Inserting this result to the first equation in (1.5.7), we determine the function \( h(x) \):
\[
(x - xy)e^{x^2/2} = \frac{\partial \varphi}{\partial x} = \frac{\partial}{\partial x} \left( -e^{x^2/2}y + h(x) \right) = -xye^{x^2/2} + h'(x),
\]
\[
(x - xy)e^{x^2/2} = -xye^{x^2/2} + h'(x), \quad h'(x) = xe^{x^2/2}, \quad h(x) = \int xe^{x^2/2} dx = e^{x^2/2} + C.
\]

Thus, finally we obtain the potential function \( \varphi(x, y) \):
\[
\varphi(x, y) = -ye^{x^2/2} + e^{x^2/2} + C = (1 - y)e^{x^2/2} + C,
\]

and the general solution of the differential equation (1.5.6)
\[
\varphi(x, y) = (1 - y)e^{x^2/2} + C = D, \quad \text{i.e.,} \quad (1 - y)e^{x^2/2} = E,
\] (1.5.8)

where \( C, D \) and \( E = D - C \) are arbitrary constants. Although we solved the modified equation (1.5.6), the solution (1.5.8) also satisfies the original differential equation (1.5.5).

If we cannot find an integrating factor that is a function of just \( x \) or just \( y \), then we must try something else. So finding an integrating factor is the most difficult part in the solution.

**Example 1.5.5 (DIFFICULT).** Solve the differential equation
\[
2y^2 - 9xy + (3xy - 6x^2)y' = 0.
\] (1.5.9)

**Answer.** **Step 1. Test for Exactness.** Let \( M(x, y) = 2y^2 - 9xy \) and \( N(x, y) = 3xy - 6x^2 \). Then
\[
\frac{\partial M}{\partial y} = 4y - 9x, \quad \text{but} \quad \frac{\partial N}{\partial x} = 3y - 12x.
\]

So the given equation is not exact.

**Step 2. Integrating Factor \( \mu(x, y) \).** We solve the equation for \( \mu(x, y) \),
\[
\frac{\partial}{\partial y} (\mu M) = \frac{\partial}{\partial x} (\mu N), \quad \frac{\partial}{\partial y} (\mu (2y^2 - 9xy)) = \frac{\partial}{\partial x} (\mu (3xy - 6x^2)),
\]
\[
(2y^2 - 9xy) \frac{\partial \mu}{\partial y} + \mu (4y - 9x) = (3xy - 6x^2) \frac{\partial \mu}{\partial x} + \mu (3y - 12x).
\] (1.5.10)

If we choose \( \mu(x, y) \) such that \( \mu_y = 0 \), then \( \mu(x, y) = \mu(x) \) and the equation becomes
\[
\mu (4y - 9x) = (3xy - 6x^2) \frac{\partial \mu}{\partial x} + \mu (3y - 12x),
\]

which is not easy to solve for \( \mu \) as just a function of \( x \).
If we choose \( \mu(x, y) \) such that \( \mu_x = 0 \), then \( \mu(x, y) = \mu(y) \) and the equation becomes

\[
(2y^2 - 9xy) \frac{\partial \mu}{\partial y} + \mu(4y - 9x) = \mu(3y - 12x),
\]

which is not easy to solve for \( \mu \) as just a function of \( y \).

So we must try something else. Let us try \( \mu(x, y) = x^a y^b \), where \( a \) and \( b \) are constants to be determined. Putting \( \mu(x, y) = x^a y^b \) into the equation \((1.5.10)\) and find the constants \( a \) and \( b \):

\[
(2y^2 - 9xy)(bx^a y^{b-1}) + (x^a y^b)(4y - 9x) = (3xy - 6x^2)(ax^{a-1} y^b) + (x^a y^b)(3y - 12x).
\]

Expanding the whole equation and dividing by \( x^a y^b \) (with assumption \( x \neq 0 \neq y \), we have

\[
2by - 9bx + 4y - 9x = 3ay - 6ax + 3y - 12x.
\]

Rearranging the terms and using that \( x \) and \( y \) are independent, we get

\[
1 + 2b - 3a = 0, \quad \text{and} \quad -3 + 9b - 6a = 0.
\]

Solving the equations for \( a \) and \( b \), we get \( a = 1 = b \), i.e., the integrating factor \( \mu(x, y) = xy \).

**Step 3. Solving Exact Equation.** Multiplying the original equation \((1.5.9)\) by \( \mu(x, y) = xy \),

\[
xy(2y^2 - 9xy) + xy(3xy - 6x^2)y' = 0, \quad 2xy^3 - 9x^2 y^2 + (3x^2 y^2 - 6x^3 y)y' = 0, \quad (1.5.11)
\]

which is an exact differential equation. A potential function \( \phi \) satisfies

\[
\frac{\partial \phi}{\partial x} = 2xy^3 - 9x^2 y^2, \quad \text{and} \quad \frac{\partial \phi}{\partial y} = 3x^2 y^2 - 6x^3 y. \quad (1.5.12)
\]

To find \( \phi \), we integrate either one. Let us integrate the first one \( \phi_x \) with respect to \( x \):

\[
\phi(x, y) = \int \left( \frac{\partial \phi}{\partial x} \right) \, dx = \int (2xy^3 - 9x^2 y^2) \, dx = x^2 y^3 - 3x^3 y^2 + g(y).
\]

Inserting this result to the second equation in \((1.5.12)\), we determine the function \( g(y) \):

\[
3x^2 y^2 - 6x^3 y = \frac{\partial \phi}{\partial y} = \frac{\partial}{\partial y} (x^2 y^3 - 3x^3 y^2 + g(y)) = 3x^2 y^2 - 6x^3 y + g'(y),
\]

\[
3x^2 y^2 - 6x^3 y = 3x^2 y^2 - 6x^3 y + g'(y), \quad g'(y) = 0, \quad g(y) = C.
\]

Thus, finally we obtain the potential function \( \phi(x, y) \):

\[
\phi(x, y) = x^2 y^3 - 3x^3 y^2 + C,
\]

and the general solution of the differential equation \((1.5.11)\)

\[
\phi(x, y) = x^2 y^3 - 3x^3 y^2 + C = D, \quad \text{i.e.,} \quad x^2 y^3 - 3x^3 y^2 = E, \quad (1.5.13)
\]

where \( C, D \) and \( E = D - C \) are arbitrary constants. Although we solved the modified equation \((1.5.11)\), the solution \((1.5.13)\) also satisfies the original differential equation \((1.5.9)\).

The method with the integrating factor may not give all the solutions. It may not give a singular solution (**Section 1.2 Separable Equations**) which is a solution but cannot be deduced from the general solution. Two examples, Example 1.5.6 and Example 1.5.7, on this case are given below. It is a peculiar case, so you may skip the examples, as long as you can understand the limitation of the integrating factor technique.
Example 1.5.6. Solve the differential equation

\[
\frac{2xy}{y-1} - y' = 0. \tag{1.5.14}
\]

**Answer.** **Particular Solution.** We observe \( y = 0 \) does satisfy the given equation. Hence, \( y = 0 \) is a particular solution.

**General Solution: Strategy for Separable Equation.** Since it is a separable equation, one can get the general solution easily by using the strategy 1.2.2 for separable equations:

\[
y - \ln|y| = x^2 + C,
\]

where \( y \neq 0 \) and \( y \neq 1 \).

**General Solution: Strategy with Integrating Factor for Exact Equation.** Let us use the integrating factor technique for an exact equation. The given equation is not exact, but \( \mu(x,y) = \frac{y-1}{y} \) is an integrating factor for \( y \neq 0 \). (We don’t have to solve a differential equation to find this integrating factor.) Multiplying the whole equation (1.5.14) by the integrating factor, we have

\[
2x - \frac{y-1}{y}y' = 0,
\]

which is exact with the potential function \( \phi(x,y) = x^2 - y + \ln|y| \). Thus the equation (1.5.14) has the general solution

\[
x^2 - y + \ln|y| = C, \quad y \neq 0.
\]

As we can see, the particular solution \( y = 0 \) cannot be obtained from the general solution. That is, \( y = 0 \) is a singular solution. □

Example 1.5.7. Solve the equation

\[
y - 3 - xy' = 0. \tag{1.5.15}
\]

**Answer.** **Particular Solution.** We observe \( y = 3 \) does satisfy the given equation. Hence, \( y = 3 \) is a particular solution.

**General Solution: Strategy for Linear Equation.** We modify the given equation,

\[
xy' - y = -3, \quad y' - \frac{1}{x}y = -\frac{3}{x}, \quad x \neq 0,
\]

which is a linear equation. So, one can get the general solution easily by using the strategy 1.3.2 for linear equations with the integrating factor \( e^{\int (-1/x)dx} = 1/x \):

\[
\left(\frac{y}{x}\right)' = -\frac{3}{x^2}, \quad \frac{y}{x} = \frac{3}{x} + C, \quad y = 3 +Cx,
\]

where \( x \neq 0 \). With \( C = 0 \), we can get the particular solution \( y = 3 \), i.e., \( y = 3 \) is not a singular solution.

**General Solution: Strategy with Integrating Factor for Exact Equation.** Let us use the integrating factor technique for an exact equation. The given equation is not exact, but \( \mu(x,y) = \frac{1}{|x(y-3)|} \) is an integrating factor for \( x \neq 0 \) and \( y \neq 3 \). (We don’t have to solve a differential equation to find this integrating factor.) Multiplying the whole equation (1.5.15) by the integrating factor, we have

\[
\frac{1}{x} - \frac{1}{y-3}y' = 0,
\]

which is exact with the potential function \( \phi(x,y) = \ln|x| - \ln|y-3| \). Thus the equation (1.5.15) has the general solution

\[
\ln|x| - \ln|y-3| = C,
\]
where $x \neq 0$ and $y \neq 3$.
The solution implicitly defined above does not give the particular solution $y = 3$. However, we can simplify
the equation and find $y$ explicitly:

$$y = 3 + Dx,$$

where $D$ is a constant. As we can see, with $D = 0$, we can get the particular solution $y = 3$. That is, $y = 3$ is
not a singular solution.

1.5.1 Separable Equations and Integrating Factors.
The separable equation

$$y' = A(x)B(y), \quad A(x)B(y) - y' = 0 \quad (1.5.16)$$
is not exact, because with $M(x,y) = A(x)B(y)$ and $N(x,y) = -1$, the partial derivatives

$$\frac{\partial M}{\partial y} = A(x)B'(y), \quad \text{and} \quad \frac{\partial N}{\partial x} = 0$$

are not equal in general.
Multiplying the whole equation (1.5.16) by the integrating factor $\mu(x,y) = 1/B(y)$ for $B(y) \neq 0$, the given
equation (1.5.16) becomes

$$A(x) - \frac{1}{B(y)} y' = 0,$$

which is exact and has the general solution

$$\int A(x) \, dx - \int \frac{1}{B(y)} \, dy = C.$$

The act of separating the variables is the same as multiplying by the integrating factor $1/B(y)$.

1.5.2 Linear Equations and Integrating Factors.
The linear equation

$$y' + p(x)y = q(x), \quad p(x)y - q(x) + y' = 0 \quad (1.5.17)$$
is not exact, because with $M(x,y) = p(x)y - q(x)$ and $N(x,y) = 1$, the partial derivatives

$$\frac{\partial M}{\partial y} = p(x), \quad \text{and} \quad \frac{\partial N}{\partial x} = 0$$

are not equal in general unless $p(x)$ is identically zero.
Multiplying the whole equation (1.5.17) by the integrating factor $\mu(x,y) = e^{\int p(x) \, dx}$, the equation (1.5.17) becomes

$$e^{\int p(x) \, dx} \left( p(x)y - q(x) \right) + e^{\int p(x) \, dx} y' = 0,$$

which is exact and has the general solution

$$e^{\int p(x) \, dx} y = \int \left( q(x)e^{\int p(x) \, dx} \right) \, dx + C, \quad y = e^{-\int p(x) \, dx} \int \left( q(x)e^{\int p(x) \, dx} \right) \, dx + Ce^{-\int p(x) \, dx}.$$

See the formula (1.3.1) in Section 1.3 Linear Differential Equations.
We deduce a formula on the integrating factor \( \mu(x,y) \) for an exact differential equation \( Mdx + Ndy = 0 \). Personally, I do not recommend that you should memorize the formulas. We start with the condition for the integrating factor \( \mu \),

\[
\frac{\partial}{\partial y}(\mu M) = \frac{\partial}{\partial x}(\mu N), \quad \mu_y + \mu M_y = \mu_x N + \mu N_x. \tag{1.5.18}
\]

**Case 1.** \( \mu(x,y) = \mu(x) \). In this case, \( \mu_y = 0 \) and the equation (1.5.18) becomes

\[
\mu M_y = \frac{d\mu}{dx} N + \mu N_x, \quad N \frac{d\mu}{dx} = \mu M_y - \mu N_x = (M_y - N_x) \mu, \quad \frac{1}{\mu} d\mu = \frac{M_y - N_x}{N} dx.
\]

If \( (M_y - N_x)/N \) is a function of only \( x \), then we can integrate both sides and get

\[
\ln \mu = \int \frac{1}{\mu} d\mu = \int \frac{M_y - N_x}{N} dx, \quad \mu(x) = \exp \left\{ \int \frac{M_y - N_x}{N} dx \right\}.
\]

**Case 2.** \( \mu(x,y) = \mu(y) \). In this case, \( \mu_x = 0 \) and the equation (1.5.18) becomes

\[
\frac{d\mu}{dy} M + \mu M_y = \mu N_x, \quad \frac{d\mu}{dy} M = \mu N_x - \mu M_y = (N_x - M_y) \mu, \quad \frac{1}{\mu} d\mu = \frac{N_x - M_y}{M} dy.
\]

If \( (N_x - M_y)/M \) is a function of only \( y \), then we can integrate both sides and get

\[
\ln \mu = \int \frac{1}{\mu} d\mu = \int \frac{N_x - M_y}{M} dy, \quad \mu(y) = \exp \left\{ \int \frac{N_x - M_y}{M} dy \right\}.
\]

**Example 1.5.8.** Find an integrating factor of the form either a function of \( x \), \( \mu(x) \), or a function of \( y \), \( \mu(y) \), which makes the differential equation \( 2xy' + 3y - 2x^3 = 0 \) to be exact.

**ANSWER.** Let us use the formula above. The equation is rewritten as \( (3y - 2x^3)dx + 2xdy = 0 \) and so letting \( M = 3y - 2x^3 \) and \( N = 2x \), we have \( M_x = -6x^2 \), \( M_y = 3 \), \( N_x = 2 \) and \( N_y = 0 \).

Since \( \frac{M_x - N_x}{N} = \frac{-6x^2}{2x} = \frac{3}{2}x \) is a function of only \( x \), the formula implies

\[
\mu(x) = \exp \left\{ \int \frac{M_y - N_x}{N} dx \right\} = \exp \left\{ \frac{1}{2} \int \frac{1}{x} dx \right\} = \exp \left\{ \frac{1}{2} \ln x \right\} = x^{1/2}.
\]

Since \( \frac{N_x - M_y}{M} = \frac{2 - 3}{-6x^2} = \frac{1}{6x^2} \) is not a function of only \( y \), we cannot use the formula. If we force to use the formula, then we have

\[
\mu(y) = \exp \left\{ \int \frac{N_x - M_y}{M} dy \right\} = \exp \left\{ \frac{1}{6} \int \frac{1}{x^2} dy \right\} = \exp \left\{ \frac{y}{6x^2} \right\},
\]

which is wrong, because it should be the function of only \( y \) (the left–hand side is \( \mu(y) \)). Hence, the integrating factor is \( \mu(x) = x^{1/2} \).
§1.6 Homogeneous, Bernoulli, and Riccati Equations.

1.6.1 Homogeneous Differential Equations.

Definition 1.6.1. A first–order differential equation is homogeneous if it has the form

\[ y' = f \left( \frac{y}{x} \right). \]  

Example 1.6.2. The following differential equations are homogeneous,

\[ y' = \frac{x}{y} \sin \left( \frac{y}{x} \right), \quad y' = \frac{y}{x} + \frac{x}{y}, \quad y' = \frac{y^2}{x^2} \frac{y}{x}. \]

Theorem 1.6.3 (Strategy). A homogeneous equation is always transformed into a separable one by the change of variable

\[ u = \frac{y}{x}, \quad i.e., \quad y = ux. \]

Differentiating it,

\[ y' = u'x + ux' = u'x + u. \]

Putting it into the equation, the given equation (1.6.1) becomes

\[ u'x + u = f(u), \quad u'x = f(u) - u, \quad x \frac{du}{dx} = f(u) - u, \]

which is separable. Using the strategy 1.2.2 for a separable equation, we get

\[ x \frac{du}{dx} = f(u) - u, \quad \frac{1}{f(u) - u} \frac{du}{dx} = \frac{1}{x} \int \frac{1}{f(u) - u} du = \int \frac{1}{x} dx = \ln |x| + C, \]

where \( f(u) - u \neq 0 \neq x \) are assumed. We compute the integral and put \( u = y/x \) back to the result and get the general solution to the equation (1.6.1).

Example 1.6.4. Solve the differential equation

\[ xy' = \frac{y^2}{x} + y, \quad x \neq 0. \]  

Answer. Step 1. Type of Equation.

1. It is easy to see that the equation (1.6.2) is neither separable nor linear.
2. The equation (1.6.2) is equivalent to

\[ \frac{y^2}{x} + y - xy' = 0, \]

which is not exact, because with \( M(x,y) = y^2/x + y \) and \( N(x,y) = -x, \)

\[ \frac{\partial M}{\partial y} = \frac{2y}{x} + 1 \neq -1 = \frac{\partial N}{\partial x}, \]

on a region in the plane.
3. Is it easy to find an integrating factor \( \mu(x,y) \) which makes the given equation to be exact?

\[ \frac{\partial (\mu M)}{\partial y} = \frac{\partial (\mu N)}{\partial x}, \quad \mu_x M + \mu M_y = \mu_x N + \mu N_x, \]

\[ \mu_y \left( \frac{y^2}{x} + y \right) + \mu \left( \frac{2y}{x} + 1 \right) = \mu_x (-x) + \mu (-1) \]

As we can see, it is not easy to find \( \mu \) satisfying the equation above.
4. We observe the given equation is equivalent to
\[ y' = \frac{y^2}{x^2} + \frac{y}{x} = \left(\frac{y}{x}\right)^2 + \frac{y}{x}, \quad (1.6.3) \]
of which the right-hand side is a function of \( y/x \). Hence, the equation (1.6.2) is homogeneous.

**Step 2. Substitution** \( u = y/x \). Since the equation is homogeneous, we use the substitution \( u = y/x \).

\[ y = ux, \quad y' = u'x + ux' = u'x + u. \]

Putting it into the equation (1.6.3), it becomes
\[ u'x + u = u^2 + u, \quad u'x = u^2, \]
which is separable. Using the strategy 1.2.2 for a separable equation, we have
\[ x \frac{du}{dx} = u^2, \quad \frac{1}{u^2} du = \frac{1}{x} dx, \quad \int \frac{1}{u^2} du = \int \frac{1}{x} dx, \quad -\frac{1}{u} = \ln|x| + C, \quad u = -\frac{1}{\ln|x| + C}. \]

Hence, by putting \( u = y/x \) back to the result, we deduce the solution of the equation (1.6.2)
\[ \frac{y}{x} = -\frac{1}{\ln|x| + C}, \quad i.e., \quad y = -\frac{x}{\ln|x| + C}. \]

**Example 1.6.5.** Solve the differential equation
\[ y' = \frac{x^2 + 3y^2}{2xy}, \quad x \neq 0 \neq y. \quad (1.6.4) \]

**Answer. Step 1. Type of Equation.** The equation is equivalent to
\[ y' = \frac{1}{2} \frac{x}{y} + \frac{3}{2} \frac{y}{x}, \quad (1.6.5) \]
which is homogeneous.

**Step 2. Substitution** \( u = y/x \). Since the equation is homogeneous, we use the substitution \( u = y/x \).

\[ y = ux, \quad y' = u'x + ux' = u'x + u. \]

Putting it into the equation (1.6.5), it becomes
\[ u'x + u = \frac{1}{2} \frac{x}{u} + \frac{3}{2} u, \quad u'x = \frac{1}{2} \frac{u}{x} + \frac{u}{2} = \frac{1 + u^2}{2u}, \]
which is separable. Using the strategy for a separable equation, we have
\[ x \frac{du}{dx} = \frac{1 + u^2}{2u}, \quad \frac{2u}{1 + u^2} du = \frac{1}{x} dx, \quad \int \frac{2u}{1 + u^2} du = \int \frac{1}{x} dx, \quad \frac{1}{1 + u^2} = \ln|x| + C, \quad u^2 = \frac{1}{\ln|x| + C} - 1, \quad u = \pm \left(\frac{1}{\ln|x| + C} - 1\right)^{1/2}. \]

Hence, by putting \( u = y/x \) back to the result, we deduce the solution of the equation (1.6.4)
\[ \frac{y}{x} = \pm \left(\frac{1}{\ln|x| + C} - 1\right)^{1/2}, \quad i.e., \quad y = \pm x \left(\frac{1}{\ln|x| + C} - 1\right)^{1/2}. \]
Before we end this subsection, let us consider the equation
\[ y' = \frac{y}{x+y}, \quad (1.6.6) \]
which is not homogeneous. But the equation can be written by
\[ y' = \frac{y}{x+y} = \frac{y/x}{1+y/x}, \quad i.e., \quad y' = \frac{y/x}{1+y/x}, \quad x \neq 0, \quad (1.6.7) \]
which is homogeneous. The solution of (1.6.7) also satisfies the equation (1.6.6). But the equation (1.6.6) may have other solutions as well. When we perform manipulations on a differential equation, we should be careful.

**Exercise 1.6.6.** Solve the homogeneous equations.

\[ y^2 + 2xy - x^2y' = 0. \]
\[ xyy' = x^2 + 2y^2. \] (Final Exam of Fall 2009)

**1.6.2 Bernoulli Differential Equation.**

**Definition 1.6.7.** A Bernoulli equation is a first–order equation
\[ y' + p(x)y = q(x)y^\alpha, \]
in which \( \alpha \) is a real number.

We observe:

1. When \( \alpha = 0 \), the Bernoulli equation becomes
\[ y' + p(x)y = q(x), \]
which is linear.
2. When \( \alpha = 1 \), the Bernoulli equation becomes
\[ y' + p(x) = q(x)y, \quad y' + (p(x) - q(x))y = 0, \]
which is both separable and linear.
3. For \( 0 \neq \alpha \neq 1 \), the Bernoulli equation is **nonlinear**.

**Theorem 1.6.8** (**Strategy**). We use the substitution (i.e., change of variable) \( v = y^{1-\alpha} \).

How does it work? We will discuss it through examples. For the rigorous and general argument, please see the Appendix 1.6.3.

**Example 1.6.9.** Solve the differential equation
\[ y' + \frac{1}{x} y = 3x^2 y^3, \quad (1.6.8) \]
which is nonlinear.

**Answer. Step 1. Substitution** \( v = y^{1-\alpha} \). Since the equation is of Bernoulli type with \( \alpha = 3 \), we use the change of variable \( v = y^{1-3} = y^{-2} \).

\[ v' = -2y^{-3}y' = -2y^{-3} \left( -\frac{1}{x}y + 3x^2y^3 \right) = \frac{2}{x}y^{-2} - 6x^2 = \frac{2}{x}v - 6x^2. \]
That is, we get
\[ v' - \frac{2}{x}v = -6x^2, \]  
which is linear.

**Step 2. Solve Transformed Equation.** We introduce the integrating factor
\[ I(x) = e^{-2\int \frac{1}{x} \, dx} = e^{-2\ln|x|} = \frac{1}{|x|^2} = \frac{1}{x^2}. \]

Multiplying the whole equation (1.6.9) by the integrating factor \( I(x) = \frac{1}{x^2} \), we get
\[
\frac{1}{x^2}v' - \frac{1}{x^2} \cdot \frac{2}{x}v = \frac{1}{x^2}(-6x^2) = -6, \quad \left( \frac{1}{x^2}v \right)' = -6, \quad \frac{1}{x^2}v = -6x + C, \quad v = Cx^2 - 6x^3.
\]

Since we are looking for the solution to the equation (1.6.8), we should change the equation back in terms of \( y \) by using \( v = y^{-2} \),
\[
y^{-2} = Cx^2 - 6x^3, \quad y = (Cx^2 - 6x^3)^{-1/2} = \frac{1}{\sqrt{Cx^2 - 6x^3}}. \]

\[ \square \]

**Example 1.6.10.** Solve the differential equation
\[ x^2 y' + 2xy - y^3 = 0, \quad x > 0, \]  
which is nonlinear.

**Answer. Step 1. Substitution** \( v = y^{1-\alpha} \). Since the equation is of Bernoulli type with \( \alpha = 3 \), we use the change of variable \( y = v^{1-3} = y^{-2} \) with \( y' = -\frac{2}{x} y + \frac{1}{x} y^3 \),
\[
v' = -2y^{-3}y' = -2y^{-3} \left( -\frac{2}{x} y + \frac{1}{x} y^3 \right) = \frac{4}{x} y^{-2} - \frac{2}{x^2} = \frac{4}{x} v - \frac{2}{x^2}, \]

That is, we get
\[ v' - \frac{4}{x^4}v = -\frac{2}{x^2}, \]  
which is linear.

**Step 2. Solve Transformed Equation.** We introduce the integrating factor
\[ I(x) = e^{-4\int \frac{1}{x} \, dx} = e^{-4\ln|x|} = \frac{1}{|x|^4} = \frac{1}{x^4}. \]

Multiplying the whole equation (1.6.11) by the integrating factor \( I(x) = \frac{1}{x^4} \), we get
\[
\frac{1}{x^4}v' - \frac{1}{x^4} \cdot \frac{4}{x^4}v = \frac{1}{x^4} \left( \frac{4}{x} v \right) = \frac{2}{x} v = \frac{2}{5x^3} + C, \quad v = \frac{2}{5x} + Cx^4.
\]

Since we are looking for the solution to the equation (1.6.10), we should change the equation back in terms of \( y \) by using \( v = y^{-2} \),
\[
y^{-2} = \frac{2}{5x} + Cx^4, \quad y = \left( \frac{2}{5x} + Cx^4 \right)^{-1/2} = \frac{1}{\sqrt{2/(5x) + Cx^4}}. \]

\[ \square \]

**Exercise 1.6.11.** Solve the Bernoulli equation,
\[ y' + (x + 1)y = e^x y^3. \]
The Bernoulli equation
\[ y' + p(x)y = q(x)y^\alpha \]  
(1.6.12)
can be solved by the change of variable \( v = y^{1-\alpha} \).

**Proof.** We assume \( \alpha \neq 0 \) and \( \alpha \neq 1 \), because when \( \alpha = 0 \) or \( \alpha = 1 \), the given equation becomes linear and it is easy to solve. (For the convenience, we use \( p \) and \( q \) for \( p(x) \) and \( q(x) \).)

**Step 1. Substitution** \( v = y^{1-\alpha} \). The change of variable implies

\[
v' = (1 - \alpha)y^{-\alpha} y' = (1 - \alpha)y^{-\alpha} (-py + qy^\alpha) = (1 - \alpha)(-py^{1-\alpha} + q) = (1 - \alpha)(-pv + q),
\]

That is, the given equation (1.6.12) turns to be

\[ v' + (1 - \alpha)pv = (1 - \alpha)q, \]
(1.6.13)
which is linear.

**Step 2. Solve Transformed Equation.** We introduce the integrating factor

\[ I(x) = e^{(1-\alpha) \int p\,dx}. \]

Multiplying the whole equation (1.6.13) by the integrating factor \( I(x) \), we get

\[
Iv' + (1 - \alpha)pvI = (1 - \alpha)qI,
\]

\[
(Iv)' = (1 - \alpha)qI, \quad Iv = (1 - \alpha) \int (qI) \, dx.
\]

Hence, the solution of the equation (1.6.13) is given by

\[
v = \frac{1 - \alpha}{I} \int (qI) \, dx = (1 - \alpha) e^{-(1-\alpha) \int p\,dx} \int \left( qe^{(1-\alpha) \int p\,dx} \right) \, dx.
\]

Since we look for the solution to the equation (1.6.12), we should express it in terms of \( y \) by using \( v = y^{1-\alpha} \),

\[
y^{1-\alpha} = (1 - \alpha) e^{-(1-\alpha) \int p\,dx} \int \left( qe^{(1-\alpha) \int p\,dx} \right) \, dx,
\]

\[
y = \left\{ (1 - \alpha) e^{-(1-\alpha) \int p(x)\,dx} \int \left( q(x)e^{(1-\alpha) \int p(x)\,dx} \right) \, dx \right\}^{\frac{1}{1-\alpha}}. \]

\[\square\]

§1.7 Applications to Mechanics, Electrical Circuits, and Orthogonal Trajectories.
Skip. Please read the textbook.

§1.8 Existence and Uniqueness for Solutions of Initial Value Problems.
Skip. Please read the textbook.
Chapter 2

Second–Order Differential Equations

§2.1 Preliminary Concepts.

Definition 2.1.1. A second–order differential equation is an equation which contains a second derivative, but no higher derivative. Generally, it has the form

$$F(x, y, y', y'') = 0,$$

where a term involving $y''$ should be appeared.

A solution of $F(x, y, y', y'') = 0$ on an interval $I$ is a function $\phi(x)$ satisfying the differential equation at each point of $I$:

$$F(x, \phi(x), \phi'(x), \phi''(x)) = 0 \quad \text{for all } x \text{ in } I.$$

Example 2.1.2. Second–order differential equations.

1. The followings are second–order differential equations:

   \[
   y'' = x^3, \quad xy'' - \cos y = e^x, \quad y'' - 4xy' + y = 2, \\
y'' = 1, \quad y'' = y, \quad x^3y'' = 5xy' + 6x^{10}y + 3.
   \]

2. $y'' + 16y = 0$ has a solution $\phi(x) = 6\cos(4x) - 17\sin(4x)$. Why? Check by putting the solution into the differential equation!

3. $x^2y'' - 5xy' + 10y = 0$ for $x > 0$ has a solution $\phi(x) = x^3 \cos(\ln x)$. Why? Check by putting the solution into the differential equation!

Definition 2.1.3. The linear second–order differential equation has the form

$$R(x)y'' + P(x)y' + Q(x)y = S(x), \quad \text{simply,} \quad y'' + p(x)y' + q(x)y = f(x), \quad (2.1.1)$$

by dividing both sides of the first equation by $R(x)$.

In this chapter, we have two main goals:

1. Study on the existence and uniqueness of the solution of the equation (2.1.1). (Throughout this course, we study only differential equations having a solution. So we do not focus on the existence.)

2. How to solve the equation (2.1.1).
§2.2 Theory of Solutions of $y'' + p(x)y' + q(x)y = f(x)$.

Let us start with the following problem from Calculus I or General Physics.

**Example 2.2.1.** A distance function $s(x)$ has the acceleration $a(x) = 2x$ with the initial velocity $v(0) = 27$ and the initial position $s(0) = 4$. Find the distance function $s(x)$.

**Answer.** One should be able to solve this kind of problem. The problem is, in fact, a second-order initial value problem: find $s(x)$ such that

$$s''(x) = 2x, \quad s(0) = 4, \quad s'(0) = 27. \tag{2.2.1}$$

Integrating it, we get

$$s'(x) = \int s''(x) \, dx = \int 2x \, dx = x^2 + C, \quad s(x) = \int s'(x) \, dx = \int (x^2 + C) \, dx = \frac{x^3}{3} + Cx + D,$$

where $C$ and $D$ are constants of integration. Applying the initial conditions to the results above,

$$27 = s'(0) = 0^2 + C, \quad C = 27,$$

$$4 = s(0) = \frac{0^3}{3} + C(0) + D = D, \quad D = 4.$$

Hence, we deduce the solution of the given second-order initial value problem (2.2.1),

$$s(x) = \frac{x^3}{3} + 27x + 4. \quad \square$$

The example says that for a second-order differential equation,

1. The solution $s(x) = \frac{x^3}{3} + Cx + D$ has two constants of integration $C$ and $D$.
2. To determine those constants, we need two initial conditions $s(0) = 4$ and $s'(0) = 27$.
3. When we have two initial conditions, we deduce only one solution $s(x) = \frac{x^3}{3} + 27x + 4$.

It is summarized in the following Theorem.

**Theorem 2.2.2 (Uniqueness of Solution).** A second-order initial value problem,

$$y'' + p(x)y' + q(x)y = f(x), \quad y(x_0) = A, \quad y'(x_0) = B,$$

has a unique (i.e., only one) solution, where $A$ and $B$ are given constants.

**Definition 2.2.3.**

1. A second-order differential equation $y'' + p(x)y' + q(x)y = 0$ is called homogeneous.
2. A second-order differential equation $y'' + p(x)y' + q(x)y = f(x)$ with $f(x) \neq 0$ is called nonhomogeneous.
3. A linear combination of two functions $f(x)$ and $g(x)$ is a function of form $af(x) + bg(x)$, where $a$ and $b$ are any constants.

One should not be confused: Homogeneous first-order differential equation is of form,

$$y' = f\left(\frac{y}{x}\right).$$

Homogeneous second-order differential equation is of form,

$$y'' + p(x)y' + q(x)y = 0.$$
**Theorem 2.2.4 (Linear Combination).** Let \( y_1(x) \) and \( y_2(x) \) be solutions of the second–order differential equation
\[
y'' + p(x)y' + q(x)y = 0
\] on an interval \( I \). Then, any linear combination of these functions is also a solution of the equation (2.2.2), that is,
\[
c_1y_1(x) + c_2y_2(x)
\] is also a solution, where \( c_1 \) and \( c_2 \) are any constants.

**Proof.** Let \( \varphi(x) = c_1y_1(x) + c_2y_2(x) \). Then,
\[
\varphi' = c_1y_1' + c_2y_2', \quad \varphi'' = c_1y_1'' + c_2y_2''.
\] So when putting them into the given equation (2.2.2), the given equation becomes
\[
\varphi'' + p(x)\varphi' + q(x)\varphi = (c_1y_1 + c_2y_2)'' + p(x)(c_1y_1 + c_2y_2)' + q(x)(c_1y_1 + c_2y_2)
\]
\[
= c_1y_1'' + c_2y_2'' + p(x)(c_1y_1' + c_2y_2') + q(x)(c_1y_1 + c_2y_2)
\]
\[
= c_1(y_1'' + p(x)y_1' + q(x)y_1) + c_2(y_2'' + p(x)y_2' + q(x)y_2)
\]
\[
= c_1(0) + c_2(0) = 0.
\] Hence, \( \varphi(x) = c_1y_1(x) + c_2y_2(x) \) is also a solution of (2.2.2). \( \square \)

**Remark 2.2.5.** We point out two remarks on the Theorem 2.2.4.

1. The Theorem does not hold for a nonhomogeneous equation.
2. The Theorem is about producing a new solution \( c_1y_1 + c_2y_2 \) from already found two solutions \( y_1 \) and \( y_2 \). How can we find those two solutions \( y_1 \) and \( y_2 \)? It will be studied in following sections. In this section, we deal with solutions already found.

**Definition 2.2.6.** Two functions \( f(x) \) and \( g(x) \) are said to be linearly dependent on an open interval \( I \), if for some constant \( c \), either \( f(x) = cg(x) \) or \( g(x) = cf(x) \) for all \( x \) in \( I \). If \( f \) and \( g \) are not linearly dependent, then we say they are linearly independent.

**Example 2.2.7.** The functions \( y_1(x) = \cos x \) and \( y_2(x) = \sin x \) are solutions of the homogeneous second–order differential equation \( y'' + y = 0 \).

(1) Verify that \( y_1 \) and \( y_2 \) are really solutions of the given differential equation.

(2) The Theorem 2.2.4 says that \( y = c_1y_1 + c_2y_2 = c_1\cos x + c_2\sin x \) with any constants \( c_1 \) and \( c_2 \) is also a solution of the given differential equation. Verify that \( y = c_1\cos x + c_2\sin x \) is really a solution.

(2) Are \( y_1 = \cos x \) and \( y_2 = \sin x \) linearly dependent or linearly independent? Justify your answer.

**Answer.** (1) We observe that for \( y_1 = \cos x \),
\[
y_1' = (\cos x)' = -\sin x, \quad y_1'' = (\cos x)'' = (-\sin x)' = -\cos x, \quad y_1'' + y_1 = -\cos x + \cos x = 0.
\] Since \( y_1 = \cos x \) satisfies the given differential equation, \( y_1 = \cos x \) is really a solution.

Similarly, for \( y_2 = \sin x \),
\[
y_2' = (\sin x)' = \cos x, \quad y_2'' = (\sin x)'' = (\cos x)' = -\sin x, \quad y_2'' + y_2 = -\sin x + \sin x = 0.
\] Since \( y_2 = \sin x \) satisfies the given differential equation, \( y_2 = \sin x \) is really a solution.

(2) For \( y = c_1\cos x + c_2\sin x \) with any constants \( c_1 \) and \( c_2 \),
\[
y' = (c_1\cos x + c_2\sin x)' = -c_1\sin x + c_2\cos x,
\]
\[
y'' = (c_1\cos x + c_2\sin x)'' = (-c_1\sin x + c_2\cos x)' = -c_1\cos x - c_2\sin x,
\]
\[ y'' + y = -c_1 \cos x - c_2 \sin x + c_1 \cos x + c_2 \sin x = 0. \]

Since \( y = c_1 \cos x + c_2 \sin x \) satisfies the given differential equation, \( y = c_1 \cos x + c_2 \sin x \) is really a solution.

(3) Suppose \( y_1 = \cos x \) and \( y_2 = \sin x \) are linear dependent. Then for some constant \( k \),
\[ y_1 = ky_2, \quad \cos x = k \sin x, \quad 1 = k \tan x, \quad \tan x = \frac{1}{k} \quad \text{for all } x. \]

It means that for any \( x \), the tangent function gives only one value \( 1/k \). That is, the tangent function is a constant function. This is obviously wrong, because the tangent function is not a constant function. Hence, the assumption is wrong and we conclude that \( y_1 = \cos x \) and \( y_2 = \sin x \) must be linearly independent. \( \square \)

As one can see, when we need to know the linear dependency/independency of given two functions, it is very painful if we use the definition. Is there a better way or method telling us whether two functions are linearly dependent or independent? Yes, it is: **Wronskian Test**. Since the test is very useful, it is very important, i.e., one must memorize it.

**Definition 2.2.8.** For two functions \( f(x) \) and \( g(x) \), the function
\[ W(x) = W(f(x), g(x)) = \begin{vmatrix} f(x) & g(x) \\ f'(x) & g'(x) \end{vmatrix} = f(x)g'(x) - f'(x)g(x) \]
is called **Wronskian** of \( f(x) \) and \( g(x) \).

**Theorem 2.2.9 (Wronskian Test).** Let \( y_1(x) \) and \( y_2(x) \) be two solutions of the differential equation \( y'' + p(x)y' + q(x)y = 0 \) on an interval \( I \). Then for the Wronskian \( W(x) = y_1(x)y_2'(x) - y_1'(x)y_2(x) \) of \( y_1 \) and \( y_2 \), we have
1. **either** \( W(x) = 0 \) for all \( x \) in \( I \) or \( W(x) \neq 0 \) for all \( x \) in \( I \).
2. \( W(x) \neq 0 \) on \( I \) if and only if \( y_1 \) and \( y_2 \) are linearly independent.

Simply speaking, the Theorem 2.2.9 says that the Wronskian is either a zero function or not. If the Wronskian is a zero function, those two functions \( y_1 \) and \( y_2 \) are linearly dependent. If the Wronskian is not a zero function, those two functions \( y_1 \) and \( y_2 \) are not linearly dependent, i.e., linearly independent.

**Example 2.2.10.** From Example 2.2.7, we recall that \( y_1 = \cos x \) and \( y_2 = \sin x \) are linearly independent. Let us use the Wronskian Test 2.2.9 to determine the linear independency:
\[ W(x) = y_1(x)y_2'(x) - y_1'(x)y_2(x) = \cos x(\sin x)' - (\cos x)\sin x = \cos^2 x + \sin^2 x = 1 \neq 0 \]
for all \( x \). Since the Wronskian \( W(x) \) is not a zero function, by the Test, we conclude that \( y_1 = \cos x \) and \( y_2 = \sin x \) are linearly independent. \( \square \)

**Theorem 2.2.11 (General Solution).** Let \( y_1 \) and \( y_2 \) be linearly independent solutions of the differential equation
\[ y'' + p(x)y' + q(x)y = 0 \quad \text{(2.2.3)} \]
on an open interval \( I \). Then every solution of the differential equation is a linear combination of \( y_1 \) and \( y_2 \), i.e., the general solution of the differential equation (2.2.3) is
\[ y(x) = c_1y_1(x) + c_2y_2(x), \]
where \( c_1 \) and \( c_2 \) are any constants.
Definition 2.2.12. Let $y_1$ and $y_2$ be solutions of the differential equation $y'' + p(x)y' + q(x)y = 0$ on an open interval $I$.

1. If $y_1$ and $y_2$ are linearly independent, $y_1$ and $y_2$ forms a fundamental set of solution on $I$.
2. If $y_1$ and $y_2$ forms a fundamental set of solution on $I$, then $c_1y_1 + c_2y_2$ is called the general solution of the given differential equation on $I$.

One will see that this definition goes to the course, LINEAR ALGEBRA (also called as CALCULUS 4 in this university). That is, one must understand and memorize it. We end this section with the proof of the Theorem 2.2.11.

Proof of Theorem 2.2.11. Let $\phi$ be any solution of the given differential equation on $I$. Then we have to show that there exist some numbers $c_1$ and $c_2$ such that $\phi = c_1y_1 + c_2y_2$.

We choose $x_0 \in I$. Let $\phi(x_0) = A$ and $\phi'(x_0) = B$. Then by Theorem 2.2.2, $\phi$ is a unique solution of the second–order initial value problem,

$$y'' + p(x)y' + q(x)y = 0, \quad y(x_0) = A, \quad y'(x_0) = B.$$ 

Now we consider the system of equations,

$$c_1y_1(x_0) + c_2y_2(x_0) = A, \quad c_1y_1'(x_0) + c_2y_2'(x_0) = B.$$ 

Solving the equations for $c_1$ and $c_2$, we get

$$c_1 = \frac{Ay_2'(x_0) - By_2(x_0)}{W(x_0)}, \quad c_2 = \frac{By_1(x_0) - Ay_1'(x_0)}{W(x_0)}, \quad (2.2.4)$$

where $W(x)$ is the Wronskian of $y_1$ and $y_2$. Then, we observe

$$c_1y_1 + c_2y_2 = \frac{Ay_2'(x_0) - By_2(x_0)}{W(x_0)}y_1 + \frac{By_1(x_0) - Ay_1'(x_0)}{W(x_0)}y_2$$

is a solution of the given differential equation. By the uniqueness of the solution in Theorem 2.2.2, we conclude

$$\phi = c_1y_1 + c_2y_2 = \frac{Ay_2'(x_0) - By_2(x_0)}{W(x_0)}y_1 + \frac{By_1(x_0) - Ay_1'(x_0)}{W(x_0)}y_2. \quad \Box$$
§2.3 Reduction of Order.
Throughout this section, we consider only the homogeneous second–order differential equation,
\[ y'' + p(x)y' + q(x)y = 0. \]  
(2.3.1)

In the previous section, we have argued on generating a general solution \( c_1y_1 + c_2y_2 \) of the homogeneous second–order differential equation \( y'' + p(x)y' + q(x)y = 0 \), when linearly independent two solutions \( y_1 \) and \( y_2 \) are known.

In this section, we develop a method of finding another solution \( y_2 \) (which is linearly independent to \( y_1 \)) of the same differential equation, when only one solution \( y_1 \) is known. So in this way, for a given one solution \( y_1 \), we can deduce a general solution.

Specifically, we look for the solution \( y_2(x) \) of the form \( u(x)y_1(x) \), i.e., \( y_2(x) = u(x)y_1(x) \). Be careful! If \( u(x) \) is a constant, then \( y_1 \) and \( y_2 \) are linearly dependent, which is not what we want. So \( u(x) \) should not be a constant.

First, we describe the strategy and then see how to use it through examples. Please do not memorize it as a formula, but try to understand how to do it.

**Goal.** When a solution \( y_1(x) \neq 0 \) is given, we want to find another solution \( y_2(x) = u(x)y_1(x) \) such that \( y_1 \) and \( y_2 \) are linearly independent.

**Theorem 2.3.1 (Strategy).** We follow the steps called the method of reduction of order.

**Step 1. Inserting** \( y_2(x) = u(x)y_1(x) \) **into Equation.** We put it into the equation (2.3.1) and simplify it.

Then we have a first–order equation,
\[ u'' + p(x)u' = 0, \quad v' + g(x)v = 0, \]
which is both separable and linear by the substitution \( v = u' \).

**Step 2. Solving Resulting Equation** \( v' + g(x)v = 0 \). It’s easy to solve the resulting equation
\[ u' = v = e^{-\int g(x)dx}, \quad u = \int e^{-\int g(x)dx}dx. \]

**Step 3. Linear Independency (Wronskian Test: Optional).** From Step 2, we have found another solution
\[ y_2(x) = u(x)y_1(x) = \left( \int e^{-\int g(x)dx}dx \right)y_1(x). \]

Using the Wronskian Test 2.2.9, we check the linear independency between \( y_1 \) and \( y_2 \). This Step is optional.

**Step 4. General Solution.** Since we have two solutions \( y_1(x) \) and \( y_2(x) = u(x)y_1(x) \) which are linearly independent, the general solution of the equation (2.3.1) is obtained by
\[ y = c_1y_1 + c_2y_2 = c_1y_1 + c_2uy_1 = (c_1 + c_2u)y_1, \quad y = (c_1 + c_2u)y_1. \]

Let us study each step above.

**Full Explanation.** **Step 1. Inserting** \( y_2(x) = u(x)y_1(x) \) **into Equation.** Since we want \( y_2(x) = u(x)y_1(x) \) to be a solution, it should satisfy the equation,
\[ y''_2 + p(x)y'_2 + q(x)y_2 = 0; \quad [u(x)y_1(x)]'' + p(x)[u(x)y_1(x)]' + q(x)[u(x)y_1(x)] = 0. \]

Expanding and simplifying it, we get
\[ 0 = (u'y_1 + uy'_1)' + p(u'y_1 + uy'_1) + qy_1 \]
\[ = u''y_1 + u'y'_1 + uy''_1 + p(u'y_1 + uy'_1) + qy_1 \]
\[ u''y_1 + 2u'y_1' + u'y_1 + pu'y_1 + pu'y_1' + qy_1 = 0, \]
\[ u( y_1'' + py_1' + qy_1 ) + u''y_1 + 2u'y_1' + pu'y_1 = 0, \]
\[ u( y_1'' + py_1' + qy_1 ) + u''y_1 + (2y_1' + py_1) u'. \]

Since \( y_1 \) is a solution, we have \( y_1'' + py_1' + qy_1 = 0 \), and so the result above becomes
\[ 0 = u''y_1 + (2y_1' + py_1) u'. \]

Since we want to find \( u \), we solve the differential equation by the change of variable \( v = u' \),
\[ 0 = v'y_1 + (2y_1' + py_1) v, \quad v' + \frac{2y_1' + py_1}{y_1} v = 0. \tag{2.3.2} \]

**Step 2. Solving Resulting Equation.** We simplify the equation (2.3.2) by letting
\[ g(x) = \frac{2y_1'(x) + p(x)y_1(x)}{y_1(x)}. \]

Then equation (2.3.2) becomes a first–order differential equation,
\[ v' + g(x) v = 0, \tag{2.3.3} \]
which is both *separable* and *linear*. Solving the equation, we have
\[ v(x) = e^{-\int g(x) \, dx}, \]
where we ignored the constant. Since we are looking for \( u \) but not \( v \), so we solve the equation
\[ u' = v = e^{-\int g(x) \, dx}, \quad u(x) = \int e^{-\int g(x) \, dx} \, dx \tag{2.3.4} \]
and finally we deduce another solution \( y_2 = uy_1 \),
\[ y_2(x) = \left( \int e^{-\int g(x) \, dx} \, dx \right) y_1(x), \quad g(x) = \frac{2y_1'(x) + p(x)y_1(x)}{y_1(x)}. \]

**Step 3. Linear Indepenency (Wronskian Test).** Let us check whether or not \( y_1 \) and \( y_2 = uy_1 \) are linearly independent. We use the Wronskian Test 2.2.9,
\[ W(x) = y_1y_2'y_2 - y_1'y_2 = y_1 (uy_1)'y_2 - y_1'u(uy_1)' = y_1 (u'y_1 + uy_1') - uy_1'y_1 = u'y_1' - u'y_1 = (e^{-\int g(x) \, dx}) y_1^2 \neq 0, \]
where the result (2.3.4) is used and the exponential function never vanishes. Since \( W(x) \neq 0 \), the Wronskian Test implies that \( y_1 \) and \( y_2 = uy_1 \) are linearly independent.

**Step 4. General Solution.** Since \( y_1 \) and \( y_2 = uy_1 \) are linearly independent solutions, by the Theorem 2.2.11, the general solution of the differential equation (2.3.1) is given by
\[ y(x) = c_1y_1 + c_2y_2 = c_1y_1 + c_2uy_1 = (c_1 + c_2u)y_1, \quad y(x) = (c_1 + c_2u(x))y_1(x), \]
where \( c_1 \) and \( c_2 \) are any constants. \( \square \)

**Remark 2.3.2.** Why is this method called the **Reduction of Order**? The big picture is as follows: for a given solution \( y_1 \), we want to find another solution \( u(x)y_1(x) \) which is linearly independent to \( y_1 \), i.e., \( u(x) \) satisfying the given requirements. In order to find such a \( u(x) \), we have to solve the equation (2.3.3), which is the first–order differential equation. For this reason, the problem on the second–order differential equation (2.3.1) turns to be the problem on the first–order differential equation (2.3.3), i.e., we reduce the order of the differential equation from order 2 to order 1.
Let us see how to use the method through examples.

**Example 2.3.3.** Suppose the homogeneous second–order differential equation \( y'' + 4y' + 4y = 0 \) has a solution \( y_1(x) = e^{-2x} \). (1) Find another solution \( y_2(x) \) which is linearly independent to \( y_1(x) \). (2) Find the general solution of the given differential equation.

**ANSWER.** **Step 1. Inserting** \( y_2(x) = u(x)y_1(x) \) **into Equation.** Let \( y_2(x) = u(x)y_1(x) = u(x)e^{-2x} \) be a solution. Then \( y_2(x) \) should satisfy the given differential equation, i.e.,

\[
0 = y_2'' + 4y_2' + 4y_2 = (ue^{-2x})'' + 4(ue^{-2x})' + 4ue^{-2x}
\]

\[
= (ue^{-2x} - 2ue^{-2x})' + 4(ue^{-2x} - 2ue^{-2x}) + 4ue^{-2x}
\]

\[
= u''e^{-2x} - 4u'e^{-2x} + 4ue^{-2x} + 4(ue^{-2x} - 2ue^{-2x}) + 4ue^{-2x} = u''e^{-2x}.
\]

Since an exponential function never vanishes, i.e., \( e^{-2x} \neq 0 \) for all \( x \), so we get \( u'' = 0 \).

**Step 2. Solving Resulting Equation.** It is easy to solve the equation

\[
u''(x) = 0, \quad u'(x) = C, \quad u(x) = Cx + D,
\]

where \( C \) and \( D \) are constants of integration. Since we need only one more solution, it is fine even if we choose \( C = 1 \) and \( D = 0 \). (Be careful! \( u(x) \) should not be constant. So we cannot choose \( C = 0 \).) Hence, we have

\[
u(x) = x, \quad y_2(x) = u(x)y_1(x) = xe^{-2x}.
\]

**Step 3. Linear Independency (Wronskian Test).** Let us check whether or not \( y_1 \) and \( y_2 = uy_1 \) are linearly independent. We use the Wronskian Test,

\[
W(x) = y_1y_2' - y_1'y_2 = e^{-2x}(xe^{-2x})' - (e^{-2x})'xe^{-2x} = e^{-4x} \neq 0,
\]

for all \( x \). Since \( W(x) \neq 0 \) for all \( x \), the Wronskian Test implies that \( y_1(x) = e^{-2x} \) and \( y_2 = u(x)y_1(x) = xe^{-2x} \) are linearly independent.

**Step 4. General Solution.** Since \( y_1(x) = e^{-2x} \) and \( y_2(x) = u(x)y_1(x) = xe^{-2x} \) are linearly independent solutions, by the Theorem 2.2.11, a general solution of the given differential equation

\[
y(x) = c_1y_1(x) + c_2y_2(x) = c_1e^{-2x} + c_2xe^{-2x} = (c_1 + c_2x)e^{-2x}.
\]

where \( c_1 \) and \( c_2 \) are any constants. \( \square \)

**Example 2.3.4.** Suppose the homogeneous second–order differential equation \( y'' - \frac{3}{x} y' + \frac{4}{x^2} y = 0 \), \( x > 0 \), has a solution \( y_1(x) = x^2 \). (1) Find another solution \( y_2(x) \) which is linearly independent to \( y_1(x) \). (2) Find the general solution of the given differential equation.

**ANSWER.** **Step 1. Inserting** \( y_2(x) = u(x)y_1(x) \) **into Equation.** Let \( y_2(x) = u(x)y_1(x) = u(x)x^2 \) be a solution. Then \( y_2(x) \) should satisfy the given differential equation, i.e.,

\[
0 = y_2'' - \frac{3}{x} y_2' + \frac{4}{x^2} y_2 = (ux^2)'' - \frac{3}{x} (ux^2)' + \frac{4}{x^2} (ux^2)
\]

\[
= (u'x^2 + 2ux)' - \frac{3}{x} (u'x^2 + 2ux) + \frac{4}{x^2} (ux^2)
\]

\[
= (u''x^2 + 4ux' + 2u) - 3(u'x + 2u) + 4u = u''x^2 + xu'.
\]

That is, we have the differential equation,

\[
x^2u'' + xu' = 0, \quad u'' + \frac{1}{x} u' = 0.
\]
The change of variable $v = u'$ implies
\[ v' + \frac{1}{x} v = 0, \]
which is both separable and linear.

**Step 2. Solving Resulting Equation.** Solving the equation, we get
\[ v = e^{-\int \frac{1}{x} dx} = e^{-\ln x} = x^{-1}, \quad u' = v = x^{-1}, \quad u(x) = \int x^{-1} dx = \ln x, \]
where we ignored the constants. (Because of the given condition $x > 0$ in the problem, $\ln |x| = \ln x$.) Hence, we have deduced
\[ u(x) = \ln x, \quad y_2(x) = u(x)y_1(x) = x^2 \ln x. \]

**Step 3. Linear Independency (Wronskian Test).** Let us check whether or not $y_1$ and $y_2 = uy_1$ are linearly independent. We use the Wronskian Test 2.2.9,
\[
W(x) = \begin{vmatrix} y_1 & y_2 \\ y'_1 & y'_2 \end{vmatrix} = \begin{vmatrix} x^2 & x^2 \ln x \\ 2x & 2x \ln x + x \end{vmatrix} = x^2 (2x \ln x + x) - x^2 \ln x (2x) = x^3 \neq 0,
\]
for all $x > 0$. (Recall the given condition in the problem.) Since $W(x) \neq 0$ for all $x > 0$, the Wronskian Test implies that $y_1(x) = x^2$ and $y_2 = u(x)y_1(x) = x^2 \ln x$ are linearly independent.

**Step 4. General Solution.** Since $y_1(x) = x^2$ and $y_2(x) = u(x)y_1(x) = x^2 \ln x$ are linearly independent solutions, by the Theorem 2.2.11, a general solution of the given differential equation
\[ y(x) = c_1y_1(x) + c_2y_2(x) = c_1x^2 + c_2x^2 \ln x = (c_1 + c_2 \ln x)x^2, \]
where $c_1$ and $c_2$ are any constants. $\square$

**Exercise 2.3.5.** Suppose the given homogeneous second–order differential equation has a solution $y_1$. (1) Find another solution $y_2$ which is linearly independent to $y_1$. (2) Find the general solution of the differential equation.
\[
\begin{array}{l}
\triangleright y'' - y' + \frac{1}{4} y = 0 \text{ with } y_1(x) = e^{x/2}.
\triangleright 2t^2 y'' + 3ty' - y = 0, \text{ with } y_1(t) = t^{-1}.
\triangleright (x^2 - x) y'' - xy' + y = 0 \text{ with } y_1(x) = x.
\triangleright x^2 y'' + xy' - 4y = 0 \text{ with } y_1(x) = x^2.
\triangleright y'' - 6y' + 9y = 0 \text{ with } y_1(x) = e^{3x}.
\triangleright (1 - x^2) y'' - 2xy' + 2y = 0 \text{ with } y_1(x) = x.
\triangleright 2x^2 y'' + xy' - 3y = 0 \text{ with } y_1(x) = 1/x. \text{ (Final Exam of Fall 2009)}
\end{array}
\]
§2.4 Constant Coefficient Homogeneous Linear Equation.
Throughout this section, we consider only the homogeneous second–order differential equation with constant coefficient,

\[ y'' + Ay' + By = 0, \]  

(2.4.1)

where \( A \) and \( B \) are real numbers.

We observe that the derivatives of an exponential function \( e^{\lambda x} \) with constant \( \lambda \) become the constant multiples of the exponential function itself, i.e., \((e^{\lambda x})' = \lambda e^{\lambda x} \) and \((e^{\lambda x})'' = \lambda^2 e^{\lambda x} \). So it could be the solution of the equation (2.4.1). For this reason, we look for the solution \( y(x) = e^{\lambda x} \).

Putting it into the equation, we have

\[ 0 = \lambda^2 e^{\lambda x} + A\lambda e^{\lambda x} + Be^{\lambda x} = e^{\lambda x} (\lambda^2 + A\lambda + B). \]

Since an exponential function can never be zero, it implies

\[ \lambda^2 + A\lambda + B = 0, \]  

(2.4.2)

which is called the characteristic equation of the differential equation (2.4.1). Solving the characteristic equation for \( \lambda \), we get

\[ \lambda = \frac{-A \pm \sqrt{A^2 - 4B}}{2}. \]

Depending on the value of \( A^2 - 4B \), we have three cases: two different real roots, repeated root, and complex roots.

Before we study each case, let us compare two equations (2.4.1) and (2.4.2). The characteristic equation is obtained by replacing \( y'' = y^{(2)}, \ y' = y^{(1)} \) and \( y = y^{(0)} \) in the equation (2.4.1) with \( \lambda^2, \ \lambda^1 \) and \( \lambda^0 = 1 \), respectively.

\[ \square \ 2.4.1 \text{ Case 1.} \ A^2 - 4B > 0. \]

In this case, the characteristic equation has two distinct real roots:

\[ a = \frac{-A + \sqrt{A^2 - 4B}}{2}, \quad b = \frac{-A - \sqrt{A^2 - 4B}}{2}, \]

and each gives the solution of the equation (2.4.1), \( y_1(x) = e^{ax} \) and \( y_2(x) = e^{bx} \), respectively. In order for \( y_1 \) and \( y_2 \) to form the fundamental set of solutions of (2.4.1), they should be linearly independent, i.e., Wronskian of \( y_1 \) and \( y_2 \) should not be zero. Let us run the Wronskian Test:

\[ W(x) = y_1(x)y_2'(x) - y_1'(x)y_2(x) = be^{ax}e^{bx} - ae^{ax}e^{bx} = e^{(a+b)x}(b-a) \neq 0, \]

because \( a \neq b \). Thus, \( y_1 \) and \( y_2 \) are linearly independent, i.e., they forms the fundamental set of solutions, i.e., the general solution of the equation (2.4.1) is given by

\[ y(x) = c_1y_1(x) + c_2y_2(x) = c_1e^{ax} + c_2e^{bx}, \]

(2.4.3)

where \( c_1 \) and \( c_2 \) are arbitrary constants.

**Example 2.4.1.** Solve the differential equation,

\[ y'' - y' - 6y = 0. \]

**Answer.** It has the characteristic equation

\[ 0 = \lambda^2 - \lambda - 6 = (\lambda + 2)(\lambda - 3), \]

of which roots are \( \lambda = -2 \) and \( \lambda = 3 \). Hence, by the formula (2.4.3), the general solution is

\[ y(x) = e^{-2x} + c_2 e^{3x}, \]

where \( c_1 \) and \( c_2 \) are arbitrary constants.
\section*{2.4.2 Case 2.} $A^2 - 4B = 0$.
The characteristic equation (2.4.2) has the repeated root $\lambda = -\frac{A}{2}$. So one solution is $y_1(x) = e^{-\frac{A}{2}x}$. Since we know one solution, we get another solution $y_2(x)$ in the form of $y_2(x) = u(x)y_1(x) = u(x)e^{-\frac{A}{2}x}$ by using the method of Reduction of Order in the previous section. Then again by the result in the previous Section 2.3 Reduction of Order, the general solution of the equation (2.4.1) is
\[
y(x) = c_1y_1(x) + c_2y_2(x) = c_1e^{-\frac{A}{2}x} + c_2u(x)e^{-\frac{A}{2}x} = (c_1 + c_2u(x))e^{-\frac{A}{2}x}.
\]

We substitute $y_2(x) = u(x)e^{-\frac{A}{2}x}$ into the differential equation (2.4.1) to find $u(x)$:
\[
0 = \left[u(x)e^{-\frac{A}{2}x}\right]'' + A\left[u(x)e^{-\frac{A}{2}x}\right]' + B\left[u(x)e^{-\frac{A}{2}x}\right]
\]
\[
= \left[u'(x)e^{-\frac{A}{2}x} - \frac{A}{2}u(x)e^{-\frac{A}{2}x}\right]' + A\left[u'(x)e^{-\frac{A}{2}x} - \frac{A}{2}u(x)e^{-\frac{A}{2}x}\right] + Bu(x)e^{-\frac{A}{2}x}
\]
\[
= u''(x)e^{-\frac{A}{2}x} - Au'(x)e^{-\frac{A}{2}x} + \frac{A^2}{4}u(x)e^{-\frac{A}{2}x} + A\left[u'(x)e^{-\frac{A}{2}x} - \frac{A}{2}u(x)e^{-\frac{A}{2}x}\right] + Bu(x)e^{-\frac{A}{2}x}
\]
\[
= u''(x)e^{-\frac{A}{2}x} - \frac{A^2}{4}u(x)e^{-\frac{A}{2}x} + Bu(x)e^{-\frac{A}{2}x}
\]
\[
= \left[u''(x) + \left(B - \frac{A^2}{4}\right)u(x)\right]e^{-\frac{A}{2}x}.
\]

Since an exponential function can never be zero, we deduce
\[
u'' + \left(B - \frac{A^2}{4}\right)v = 0.
\]

However, we are discussing the case $A^2 - 4B = 0$, so the equation is in fact
\[
u'' = 0, \quad v'(x) = C, \quad v(x) = Cx + D,
\]
and we choose $C = 1$ and $D = 0$. (Why? See the Section 2.3 Reduction of Order.) That is, $u(x) = x$ and we get $y_2(x) = xe^{-\frac{A}{2}x}$ and the general solution of the equation (2.4.1),
\[
y(x) = c_1y_1(x) + c_2y_2(x) = (c_1 + c_2u(x))e^{-\frac{A}{2}x} = (c_1 + c_2x)e^{-\frac{A}{2}x}.
\]

\section*{Example 2.4.2.} Solve the differential equation,
\[
y'' - 6y' + 9y = 0.
\]

\textbf{Answer.} It has the characteristic equation
\[
0 = \lambda^2 - 6\lambda + 9 = (\lambda - 3)^2,
\]
which has the repeated root, $\lambda = 3$. Hence, by the formula (2.4.4), the general solution is
\[
y(x) = (c_1 + c_2x)e^{3x},
\]
where $c_1$ and $c_2$ are arbitrary constants.
Case 3. $A^2 - 4B < 0$.

A complex number $i$ is defined by $i = \sqrt{-1}$ so that $i^2 = -1$. If $A^2 - 4B < 0$, then $4B - A^2 > 0$ and we have

$$\sqrt{A^2 - 4B} = \sqrt{(-1) (4B - A^2)} = \sqrt{2^2 (4B - A^2)} = i\sqrt{4B - A^2}.$$ 

So in the case of $A^2 - 4B < 0$, the characteristic equation has the complex roots

$$\lambda = \frac{-A \pm i\sqrt{4B - A^2}}{2}.$$ 

For convenience, we write

$$p = \frac{A}{2}, \quad \text{and} \quad q = \frac{\sqrt{4B - A^2}}{2}.$$ 

Then, the roots become

$$\lambda = \frac{-A \pm i\sqrt{4B - A^2}}{2} = \frac{-A}{2} \pm \frac{i\sqrt{4B - A^2}}{2} = p \pm iq$$

which yields two solutions

$$y_1(x) = e^{(p+iq)x}, \quad y_2(x) = e^{(p-iq)x}.$$ 

Now we check the linear independency of $y_1$ and $y_2$ via the Wronskian Test:

$$W(x) = y_1(x)y_2'(x) - y_1'(x)y_2(x) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} = \begin{vmatrix} e^{(p+iq)x} & e^{(p-iq)x} \\ (p+iq)e^{(p+iq)x} & (p-iq)e^{(p-iq)x} \end{vmatrix} = (p - iq)e^{(p+iq)x}e^{(p-iq)x} - (p + iq)e^{(p+iq)x}e^{(p-iq)x} = e^{2px}(-2iq).$$ 

The condition $A^2 - 4B < 0$ implies $q = \sqrt{4B - A^2} \neq 0$. That is, $W(x) \neq 0$ for all $x$ and so $y_1$ and $y_2$ are linearly independent and thus the general solution of the equation (2.4.1) is given by

$$y(x) = c_1y_1(x) + c_2y_2(x) = c_1e^{(p+iq)x} + c_2e^{(p-iq)x}. \quad (2.4.5)$$ 

Example 2.4.3. Solve the differential equation,

$$y'' + 2y' + 6y = 0.$$ 

**ANSWER.** It has the characteristic equation

$$\lambda^2 + 2\lambda + 6 = 0,$$

which has the complex roots

$$\lambda = -1 \pm i\sqrt{5}.$$ 

Hence, by the formula (2.4.5), the general solution is

$$y(x) = c_1e^{-1+i\sqrt{5}}x + c_2e^{-1-i\sqrt{5}}x,$$

where $c_1$ and $c_2$ are arbitrary constants.
\[ 2.4.4 \text{ Alternative General Solution in Case of Complex Roots.} \]

Let us introduce Euler’s Formula:

\[ e^{ix} = \cos x + i \sin x. \]

It implies

\[ e^{-ix} = \cos (-x) + i \sin (-x) = \cos x - i \sin x. \]

When we use the formula, the general solution (2.4.5) becomes

\[
y(x) = c_1 e^{(p+iq)x} + c_2 e^{(p-iq)x} = c_1 e^{px+iqx} + c_2 e^{px-iqux} = c_1 e^{px} e^{iqx} + c_2 e^{px} e^{-iqx} \\
= e^{px} (c_1 e^{iqx} + c_2 e^{-iqx}) = e^{px} \{c_1 [\cos (qx) + i \sin (qx)] + c_2 [\cos (qx) - i \sin (qx)]\} \\
= e^{px} [c_1 + c_2] \cos (qx) + i [c_1 - c_2] \sin (qx)] \\
= (c_1 + c_2) e^{px} \cos (qx) + i (c_1 - c_2) e^{px} \sin (qx). \]

If we choose \( c_1 = c_2 = 1/2 \), we obtain one solution \( y_3(x) = e^{px} \cos (qx) \).

If we choose \( c_1 = 1/(2i) \) and \( c_2 = -1/(2i) \), we obtain another solution \( y_4(x) = e^{px} \sin (qx) \).

The Wronskian Test on \( y_3 \) and \( y_4 \) implies

\[ W(x) = y_3 y_4' - y_3' y_4 = e^{2px} \neq 0, \]

i.e., they are linearly independent, i.e., the general solution of the equation (2.4.1) is their linear combination,

\[ y(x) = c_3 y_3(x) + c_4 y_4(x) = c_3 e^{px} \cos (qx) + c_4 e^{px} \sin (qx) = e^{px} (c_3 \cos (qx) + c_4 \sin (qx)), \quad (2.4.6) \]

where \( c_3 \) and \( c_4 \) are arbitrary constants.

Example 2.4.4. Solve the initial value problem,

\[ y'' - 4y' + 53y = 0, \quad y(\pi) = -3, \quad y'(\pi) = 2. \]

Answer 1. Using (2.4.6). The differential equation has the characteristic equation,

\[ \lambda^2 - 4\lambda + 53 = 0, \]

which has the complex roots \( \lambda = 2 \pm 7i = p \pm iq \). So by the formula (2.4.6), we have the general solution of the differential equation,

\[ y(x) = e^{px} (c_1 \cos (qx) + c_2 \sin (qx)) = e^{2x} (c_1 \cos (7x) + c_2 \sin (7x)). \]

The initial conditions say

\[
\begin{align*}
-3 &= y(\pi) = e^{2\pi} (c_1 \cos (7\pi) + c_2 \sin (7\pi)) = e^{2\pi} (c_1 (-1) + c_2 (0)) = -c_1 e^{2\pi}, \\
2 &= y'(\pi) = 2e^{2\pi} (c_1 \cos (7\pi) + c_2 \sin (7\pi)) + e^{2\pi} (-7c_1 \sin (7\pi) + 7c_2 \cos (7\pi)) \\
&= 2e^{2\pi} (c_1 (-1) + c_2 (0)) + e^{2\pi} (-7c_1 (0) + 7c_2 (-1)) = -e^{2\pi} (2c_1 + 7c_2),
\end{align*}
\]

where Euler’s formula is used,

\[
\begin{align*}
e^{7\pi i} &= \cos (7\pi) + i \sin (7\pi) = \cos \pi + i \sin \pi = -1, \\
e^{-7\pi i} &= \cos (7\pi) - i \sin (7\pi) = \cos \pi - i \sin \pi = 1.
\end{align*}
\]

That is, \( e^{(2+7\pi)i} = -e^{2\pi} = e^{(2-7\pi)i} \).

Solving two equation coming from the initial conditions, we get \( c_1 = 3e^{-2\pi} \) and

\[
2 = -e^{2\pi} [2(3e^{-2\pi}) + 7c_2], \quad 2 = -6 - 7c_2 e^{2\pi}, \quad c_2 = -\frac{8}{7} e^{-2\pi}.
\]
Therefore, we conclude that the solution of the initial value problem is
\[
y(x) = e^{2x} (c_1 \cos(7x) + c_2 \sin(7x)) = e^{2x} \left(3e^{-2\pi} \cos(7x) - \frac{8}{7}e^{-2\pi} \sin(7x)\right)
\]
\[
= e^{2x-2\pi} \left(3 \cos(7x) - \frac{8}{7} \sin(7x)\right) = e^{2(x-\pi)} \left(3 \cos(7x) - \frac{8}{7} \sin(7x)\right).
\]

\[\square\]

**Answer 2. Using (2.4.5).** The differential equation has the characteristic equation,
\[
\lambda^2 - 4\lambda + 53 = 0,
\]
which has the complex roots \(\lambda = 2 \pm 7i = p \pm iq\). So by the formula (2.4.5), we have the general solution of the differential equation,
\[
y(x) = c_1 e^{(p+iq)x} + c_2 e^{(p-iq)x} = c_1 e^{(2+7i)x} + c_2 e^{(2-7i)x}.
\]
The initial conditions say
\[
-3 = y(\pi) = c_1 e^{(2+7i)\pi} + c_2 e^{(2-7i)\pi}, \quad 2 = y'(\pi) = c_1 (2+7i) e^{(2+7i)\pi} + c_2 (2-7i) e^{(2-7i)\pi}.
\]
Using Euler’s formula,
\[
e^{7\pi i} = \cos(7\pi) + i \sin(7\pi) = \cos \pi + i \sin \pi = -1,
\]
\[
e^{-7\pi i} = \cos(7\pi) - i \sin(7\pi) = \cos \pi - i \sin \pi = -1.
\]
That is, \(e^{(2+7i)\pi} = -e^{2\pi} = e^{(2-7i)\pi}\). It allows us to simplify the equations coming from the initial conditions,
\[
-3 = c_1 e^{(2+7i)\pi} + c_2 e^{(2-7i)\pi} = -e^{2\pi} (c_1 + c_2), \quad i.e., \quad c_1 + c_2 = 3e^{-2\pi},
\]
\[
2 = c_1 (2+7i) e^{(2+7i)\pi} + c_2 (2-7i) e^{(2-7i)\pi} = -e^{2\pi} [c_1 (2+7i) + c_2 (2-7i)]
\]
\[
= -e^{2\pi} [2 (c_1 + c_2) + 7i (c_1 - c_2)].
\]
We have two equations and two unknowns \(c_1\) and \(c_2\). So we can solve the system of equations for the unknowns \(c_1\) and \(c_2\). Solving the system of equations, we have
\[
2 = -6 - 7ie^{2\pi} (c_1 - c_2), \quad c_1 - c_2 = -\frac{8}{7}ie^{-2\pi} = i \frac{8}{7} e^{-2\pi},
\]
where the identity \(-1/i = i\) is used. Finally, we deduce
\[
c_1 = \frac{1}{2} \left(3e^{-2\pi} + i \frac{8}{7}e^{-2\pi}\right) = \frac{e^{-2\pi}}{2} \left(3 + i \frac{8}{7}\right), \quad c_2 = \frac{1}{2} \left(3e^{-2\pi} - i \frac{8}{7}e^{-2\pi}\right) = \frac{e^{-2\pi}}{2} \left(3 - i \frac{8}{7}\right).
\]
Therefore, the solution of the initial value problem is
\[
y(x) = c_1 e^{(2+7i)x} + c_2 e^{(2-7i)x} = \frac{e^{-2\pi}}{2} \left(3 + i \frac{8}{7}\right) e^{(2+7i)x} + \frac{e^{-2\pi}}{2} \left(3 - i \frac{8}{7}\right) e^{(2-7i)x}
\]
\[
= \frac{e^{-2\pi}}{2} \left[\left(3 + i \frac{8}{7}\right) e^{(2+7i)x} + \left(3 - i \frac{8}{7}\right) e^{(2-7i)x}\right].
\]
\[\square\]

We have given two answers above. Are those two answers same? That is,
\[
\text{Answer 1 } = e^{2(x-\pi)} \left(3 \cos(7x) - \frac{8}{7} \sin(7x)\right)
\]
We will expand and simplify the right-hand side and obtain the left-hand side.

(\text{Answer 2.4.5.})

The example above suggests the following preference for a differential equation in the complex root case. If the problem has the initial conditions, i.e., for the initial value problem in the complex roots case, it is recommended that the general solution (2.4.6) should be used.

If the problem does not have a initial condition, i.e., when we are asked to find only the general solution, one may use either (2.4.5) or (2.4.6). However, personally, I would recommend the second one (2.4.6).

\textbf{Exercise 2.4.5.} Solve the differential equations.

\begin{itemize}
  \item $y'' + y' - 2y = 0$ with $y(0) = 4$ and $y'(0) = -5$.
  \item $y'' + y' + \frac{1}{4}y = 0$ with $y(0) = 3$ and $y'(0) = -7/2$.
  \item $y'' + 0.4y' + 9.04y = 0$ with $y(0) = 0$ and $y'(0) = 3$.
  \item $y'' - 6y' - 7y = 0$.
  \item $4y'' - 20y' + 25y = 0$.
  \item $y'' + 2y' + 5y = 0$.
  \item $y'' + 5y' + 6y = 0$ with $y(0) = 2$ and $y'(0) = 3$. (Final Exam of Fall 2009)
\end{itemize}

\textbf{Remark 2.4.6 (SUMMARY).} Solve the constant coefficient second–order differential equation $y'' + Ay' + By = 0$, where $A$ and $B$ are real numbers. We find and solve the characteristic equation

$$\lambda^2 + A\lambda + B = 0, \quad \lambda = \frac{-A \pm \sqrt{A^2 - 4B}}{2}.$$

\textbf{Case 1.} If the characteristic equation has two distinct real roots $\lambda = a$ and $\lambda = b$, then the general solution of the differential equation is

$$y = c_1 e^{ax} + c_2 e^{bx}.$$

\textbf{Case 2.} If the characteristic equation has repeated roots $\lambda = a$, then the general solution of the differential equation is

$$y = c_1 e^{ax} + c_2 xe^{ax} = (c_1 + c_2 x) e^{ax}.$$

\textbf{Case 3.} If the characteristic equation has complex roots $\lambda = p \pm qi$, then the general solution of the differential equation is

$$y = e^{px} (c_1 \cos(qx) + c_2 \sin(qx)) \quad \text{or} \quad y = c_1 e^{(p+qi)x} + c_2 e^{(p-qi)x}.$$
Now we develop amazing identities from Euler’s formula:

\[ e^{7ix} = \cos(7x) + i\sin(7x), \quad e^{-7ix} = \cos(7x) - i\sin(7x). \]

Adding/Subtracting those two equations, we get

\[ e^{7ix} + e^{-7ix} = 2\cos(7x), \quad e^{7ix} - e^{-7ix} = 2i\sin(7x). \]

**Theorem 2.4.7 (Identities on Exponential and Trigonometric Functions).** For any real \( x \),

\[
\cos x = \frac{e^{ix} + e^{-ix}}{2}, \quad \sin x = \frac{e^{ix} - e^{-ix}}{2i}
\]

By the Theorem, we have

\[
\text{Answer 2} = e^{2\pi - 2\pi/2} \left[ 3 + i\frac{8}{7} (e^{(2+7i)x} + e^{(2-7i)x}) \right]
= e^{2\pi} \left[ 3 (e^{7ix} + e^{-7ix}) + i\frac{8}{7} (e^{7ix} - e^{-7ix}) \right]
= e^{2\pi} \left[ 6\cos(7x) + i\frac{8}{7} 2i\sin(7x) \right]
= e^{2\pi} \left[ 6\cos(7x) - \frac{16}{7} \sin(7x) \right] \quad \text{(because } i^2 = -1)\]
= \left[ 3\cos(7x) - \frac{8}{7} \sin(7x) \right] = \text{Answer 1}
§2.5 Euler’s Equation.

Throughout this section, we consider only the homogeneous second–order differential equation with certain coefficient,
\[ y'' + \frac{A}{x} y' + \frac{B}{x^2} y = 0, \quad \text{or} \quad x^2 y'' + Axy' + By = 0 \quad (x > 0) \tag{2.5.1} \]
where \( A \) and \( B \) are real numbers. The equation is called Euler’s equation.

Since the domain of \( y(x) \) is positive real numbers \((x > 0)\), we can use the change of variable,
\[ x(t) = e^t, \quad \text{or equivalently} \quad t(x) = \ln x, \]
and so \( y(x) = y(x(t)) = (y \circ x)(t) \). Let \( Y(t) = (y \circ x)(t) \). Then by the Chain Rule in Calculus I,
\[ Y'(t) = \frac{dY(t)}{dt} = \frac{d}{dt} (y \circ x)(t) = \frac{d}{dx} (y \circ x)(t) \frac{dx}{dt} = \frac{d}{dx} y(x) \frac{dx}{dt} = y'(x) \frac{dx}{dt} = y'(x) e^t = y'(x) x, \]
\[ Y''(t) = \frac{dY'(t)}{dt} = \frac{d}{dt} (y'(x) x) = \frac{d}{dx} (y'(x) x) \frac{dx}{dt} = (y''(x)x + y'(x)) e^t = (y''(x)x + y'(x)) x = y''(x)x^2 + y'(x)x. \]
From those results, we deduce
\[ xy'(x) = Y'(t), \quad x^2 y''(x) = Y''(t) - xy'(x) = Y''(t) - Y'(t). \]
Hence, Euler’s equation (2.5.1) becomes in terms of \( Y(t) \),
\[ Y''(t) - Y'(t) + AY'(t) + BY(t) = 0, \quad Y''(t) + (A - 1)Y'(t) + BY(t) = 0, \tag{2.5.2} \]
which is a constant coefficient differential equation for \( Y(t) \).

In the previous Section 2.4 Constant Coefficient Homogeneous Linear Equation, we have studied on how to solve the equation (2.5.2). We solve the equation (2.5.2) and then later we put \( x = e^t \) or \( t = \ln x \) back to get the solution of (2.5.2) in terms of \( x \). In carrying out the strategy, it is useful to recall that for \( x > 0 \) and any real number \( b \),
\[ x^b = e^{b \ln x}. \]

Remark 2.5.1. What should we memorize? We should memorize the equation (2.5.2). Please do not forget that we have to express the solution of the equation (2.5.2) in terms of \( x \), because the original equation (2.5.1) is given in terms of \( x \).

Remark 2.5.2 (Another Strategy). There is another strategy for Euler’s equation. Please see the Appendix 2.5.1, which I personally recommend, although we will follow the strategy developed above, because it is given in the textbook.

Example 2.5.3. Solve the Euler equation
\[ y'' + \frac{2}{x} y' - \frac{6}{x^2} y = 0, \quad i.e., \quad x^2 y'' + 2xy' - 6y = 0. \]

Answer. Let \( x = e^t \). Then by the change of variable \( x = e^t \) or \( t = \ln x \), the equation becomes
\[ Y''(t) + (2 - 1)Y'(t) - 6Y(t) = 0, \quad i.e., \quad Y''(t) + Y'(t) - 6Y(t) = 0, \tag{2.5.3} \]
which is a constant coefficient homogeneous linear differential equation. We use the technique in the previous section 2.4. It has the characteristic equation
\[ 0 = \lambda^2 + \lambda - 6 = (\lambda - 2)(\lambda + 3), \]
which has two distinct real roots $\lambda = 2$ and $\lambda = -3$. So the equation (2.5.3) has the general solution
\[ Y(t) = c_1 e^{2t} + c_2 e^{-3t}, \]
where $c_1$ and $c_2$ are arbitrary constants. Now using the change of variable $(x = e^t$ or $t = \ln x)$ again, we deduce the solution of the original equation
\[ y(x) = c_1 x^2 + c_2 x^{-3}, \quad (x > 0) \]
because $e^{2t} = (e^t)^2 = x^2$ and $e^{-3t} = (e^t)^{-3} = x^{-3}$ or $e^{2t} = e^{2\ln x} = x^2$ and $e^{-3t} = e^{-3\ln x} = x^{-3}$. \[\square\]

**Example 2.5.4.** Solve the Euler equation
\[ y'' - \frac{5}{x} y' + \frac{9}{x^2} y = 0, \quad \text{i.e.,} \quad x^2 y'' - 5xy' + 9y = 0. \]

**Answer.** Let $x = e^t$. Then by the change of variable $x = e^t$ or $t = \ln x$, the equation becomes
\[ Y''(t) + (-5 - 1) Y'(t) + 9Y(t) = 0, \quad \text{i.e.,} \quad Y''(t) - 6Y'(t) + 9Y(t) = 0, \quad (2.5.4) \]
which is a constant coefficient homogeneous linear differential equation. We use the technique in the previous section 2.4. It has the characteristic equation
\[ 0 = \lambda^2 - 6\lambda + 9 = (\lambda - 3)^2, \]
which has repeated root $\lambda = 3$. So the equation (2.5.4) has the general solution
\[ Y(t) = (c_1 + c_2 t) e^{3t}, \]
where $c_1$ and $c_2$ are arbitrary constants. Now using the change of variable $(x = e^t$ or $t = \ln x)$ again, we deduce the solution of the original equation
\[ y(x) = (c_1 + c_2 \ln x) x^3, \quad (x > 0) \]
because $e^{3t} = (e^t)^3 = x^3$ or $e^{3t} = e^{3\ln x} = x^3$. \[\square\]

**Example 2.5.5.** Solve the Euler equation
\[ y'' + \frac{3}{x} y' + \frac{10}{x^2} y = 0, \quad \text{i.e.,} \quad x^2 y'' + 3xy' + 10y = 0. \]

**Answer.** Let $x = e^t$. Then by the change of variable $x = e^t$ or $t = \ln x$, the equation becomes
\[ Y''(t) + (3 - 1) Y'(t) + 10Y(t) = 0, \quad \text{i.e.,} \quad Y''(t) + 2Y'(t) + 10Y(t) = 0, \quad (2.5.5) \]
which is a constant coefficient homogeneous linear differential equation. We use the technique in the previous section 2.4. It has the characteristic equation
\[ \lambda^2 + 2\lambda + 10 = 0, \]
which has complex roots
\[ \lambda = -1 \pm i\sqrt{3}. \]
So the equation (2.5.5) has the general solution
\[ Y(t) = e^{-t} \left( c_1 \cos (3t) + c_2 \sin (3t) \right), \]
where $c_1$ and $c_2$ are arbitrary constants. Now using the change of variable $(x = e^t$ or $t = \ln x)$ again, we deduce the solution of the original equation
\[ y(x) = x^{-1} \left( c_1 \cos (3 \ln x) + c_2 \sin (3 \ln x) \right), \quad (x > 0) \]
because $e^{-t} = (e^t)^{-1} = x^{-1}$ or $e^{-t} = e^{-\ln x} = x^{-1}$. \[\square\]
As usual, we can solve an initial value problem by finding the general solution of the differential equation and then finding the constants satisfying the initial conditions.

**Example 2.5.6.** Solve the initial value problem

\[ x^2y'' - 5xy' + 10y = 0, \quad y(1) = 4, \ y'(1) = -6. \]

**Answer.** We observe that the given differential equation is Euler’s equation. Let \( x = e^t \). Then by the change of variable \( x = e^t \) or \( t = \ln x \), the equation becomes

\[ Y''(t) + (-5 - 1)Y'(t) + 10Y(t) = 0, \quad \text{i.e.,} \quad Y''(t) - 6Y'(t) + 10Y(t) = 0, \quad (2.5.6) \]

which is a constant coefficient homogeneous linear differential equation. We use the technique in the previous section 2.4. It has the characteristic equation

\[ \lambda^2 - 6\lambda + 10 = 0, \]

which has complex roots

\[ \lambda = 3 \pm i. \]

So the equation (2.5.6) has the general solution

\[ Y(t) = e^{3t}(c_1 \cos t + c_2 \sin t), \]

where \( c_1 \) and \( c_2 \) are arbitrary constants. Now using the change of variable \( (x = e^t \) or \( t = \ln x \) again, we deduce the solution of the original equation

\[ y(x) = x^3(c_1 \cos(\ln x) + c_2 \sin(\ln x)), \quad (x > 0) \]

because \( e^{3t} = (e^t)^3 = x^3 \) or \( e^{3\ln x} = x^3 \).

Now we use the initial conditions to determine the constants \( c_1 \) and \( c_2 \). \( y(1) = 4 \) implies

\[ 4 = y(1) = 1^3(c_1 \cos(\ln 1) + c_2 \sin(\ln 1)) = c_1 \cos 0 + c_2 \sin 0 = c_1, \quad c_1 = 4. \]

The solution becomes

\[
\begin{align*}
y(x) & = x^3(4\cos(\ln x) + c_2 \sin(\ln x)) \\
y'(x) & = 3x^2(4\cos(\ln x) + c_2 \sin(\ln x)) + x^3 \left[-4\sin(\ln x) \frac{1}{x} + c_2 \cos(\ln x) \frac{1}{x}\right] \\
& = 3x^2(4\cos(\ln x) + c_2 \sin(\ln x)) + x^2(-4\sin(\ln x) + c_2 \cos(\ln x)) \\
& = (12 + c_2)x^2\cos(\ln x) + (3c_2 - 4)x^2\sin(\ln x).
\end{align*}
\]

The other initial condition \( y'(1) = -6 \) yields

\[ -6 = y'(1) = (12 + c_2)\cos(\ln 1) + (3c_2 - 4)\sin(\ln 1) = 12 + c_2, \quad c_2 = -18. \]

Therefore, finally, we get the solution of the initial value problem

\[ y(x) = x^3(4\cos(\ln x) - 18\sin(\ln x)) = 2x^3(2\cos(\ln x) - 9\sin(\ln x)), \quad (x > 0) \]

**Remark 2.5.7** (IMPORTANT OBSERVATION). Let us observe the structure of the solutions.

1. Constant coefficient linear equation (Section 2.4)

\[ y'' + Ay' + By = 0 \quad (2.5.7) \]

has the solution of form

\[ e^{\lambda x}, \quad xe^{\lambda x}, \quad e^{\lambda x} \cos(\beta x), \quad e^{\lambda x} \sin(\beta x). \]
2. Euler’s equation (This Section)

\[ y'' + \frac{A}{x} y' + \frac{B}{x^2} y = 0 \quad \text{or} \quad x^2y'' + Axy' + By = 0 \]  

(2.5.8)

has the solution of form

\[ x^r, \quad x^r \ln x, \quad x^p \cos (q \ln x), \quad x^p \sin (q \ln x). \]

By the observation above, we can say \( x^3 \) cannot be a solution of the equation \((2.5.7)\) and \( e^{6x} \) cannot be a solution of Euler’s equation \((2.5.8)\).

Exercise 2.5.8. Solve the differential equation \( x^2y'' - 2y = 0, x > 0 \). (Final Exam of Fall 2009)

Remark 2.5.9 (SUMMARY). Solve the Euler equation

\[ y'' + \frac{A}{x} y' + \frac{B}{x^2} y = 0, \quad \text{or} \quad x^2y'' + Axy' + By = 0 \quad (x > 0) \]

where \( A \) and \( B \) are real numbers.

By using the technique in the previous section 2.4, we solve the constant coefficient equation

\[ Y''(t) + (A - 1) Y'(t) + BY(t) = 0. \]

Once we have the general solution, we express it in terms of \( x \) by using \( x = e^t \) or \( t = \ln x \).

\[ \Box \]

2.5.1 Appendix.

Part I. Basic Transformation

We introduce another strategy for Euler’s equation

\[ y'' + \frac{A}{x} y' + \frac{B}{x^2} y = 0, \quad \text{or} \quad x^2y'' + Axy' + By = 0 \quad (x > 0) \]  

(2.5.9)

where \( A \) and \( B \) are real numbers.

We look for the solution in the form of \( y(x) = x^\lambda \), where \( \lambda \) is a constant to be determined. Putting it into the equation \((2.5.9)\), we get

\[ 0 = x^2 \left( x^\lambda \right)'' + Ax \left( x^\lambda \right)' + Bx^\lambda = x^2 \left( \lambda (\lambda - 1) x^{\lambda - 2} \right) + Ax\lambda x^{\lambda - 1} + Bx^\lambda \]

\[ = x^\lambda (\lambda (\lambda - 1) + A\lambda + B) = x^\lambda (\lambda^2 + (A - 1) \lambda + B). \]

Since \( x > 0 \), so \( x^\lambda > 0 \) and thus we have the auxiliary equation

\[ \lambda^2 + (A - 1) \lambda + B = 0, \]  

(2.5.10)

which has roots

\[ \lambda = \frac{-A + 1 \pm \sqrt{(A - 1)^2 - 4B}}{2} = \frac{1 - A}{2} \pm \sqrt{\frac{1}{4} (1 - A)^2 - B}. \]
Case 1. Two Real Distinct Roots

Suppose the equation (2.5.10) has two real distinct roots

\[ a = \frac{1-A}{2} + \sqrt{\frac{1}{4} (1-A)^2 - B}, \quad b = \frac{1-A}{2} - \sqrt{\frac{1}{4} (1-A)^2 - B}. \]

In this case, solutions of (2.5.9) are

\[ y_1(x) = x^a, \quad y_2(x) = x^b. \]

They are linearly independent through the Wronskian Test. Thus, the general solution of (2.5.9) is

\[ y(x) = c_1y_1(x) + c_2y_2(x) = c_1x^a + c_2x^b, \]  \hspace{1cm} (2.5.11)

where \( c_1 \) and \( c_2 \) are arbitrary constants.

**Example 2.5.10.** Solve the differential equation

\[ y'' + \frac{3}{2x} y' - \frac{1}{2x^2} y = 0, \quad \text{or} \quad x^2 y'' + \frac{3}{2} x y' - \frac{1}{2} y = 0. \quad (x > 0) \]

**ANSWER.** From the given equation, we deduce the equation on the power of \( x^l \) like (2.5.10)

\[ \lambda^2 + \left(\frac{3}{2} - 1\right) \lambda - \frac{1}{2} = 0, \quad 0 = 2\lambda^2 + \lambda - 1 = (\lambda + 1)(2\lambda - 1), \quad \lambda = -1, \lambda = \frac{1}{2}. \]

By the formula (2.5.11), we have the general solution

\[ y(x) = c_1x^{-1} + c_2x^{1/2}, \]

where \( c_1 \) and \( c_2 \) are arbitrary constants.

Case 2. Repeated Root

Suppose \((1-A)^2 - 4B = 0\) in the equation (2.5.10). Then the equation has repeated root

\[ \lambda = \frac{1-A}{2}. \]

In this case, one solution of (2.5.9) is

\[ y_1(x) = x^{\frac{1-A}{2}}. \]

To get another solution \( y_2(x) = u(x)y_1(x) = u(x)x^{\frac{1-A}{2}} \) which is linearly dependent to \( y_1(x) \), we use the method of Reduction of Order in Section 2.3. Then we obtain another solution

\[ y_2(x) = u(x)x^{\frac{1-A}{2}} = (\ln x)x^{\frac{1-A}{2}}. \]

(How? Why? Check/Deduce \( y_2(x) \) by yourselves.) Therefore, the general solution of (2.5.9) is

\[ y(x) = c_1y_1(x) + c_2y_2(x) = c_1x^{\frac{1-A}{2}} + c_2(\ln x)x^{\frac{1-A}{2}} = (c_1 + c_2 \ln x)x^{\frac{1-A}{2}}. \]  \hspace{1cm} (2.5.12)

where \( c_1 \) and \( c_2 \) are arbitrary constants.

**Example 2.5.11.** Solve the differential equation

\[ y'' - \frac{5}{x} y' + \frac{9}{x^2} y = 0, \quad \text{or} \quad x^2 y'' - 5xy' + 9y = 0. \quad (x > 0) \]

**ANSWER.** From the given equation, we deduce the equation on the power of \( x^l \) like (2.5.10)

\[ \lambda^2 + (-5 - 1) \lambda + 9 = 0, \quad 0 = \lambda^2 - 6\lambda + 9 = (\lambda - 3)^2, \quad \lambda = 3. \]

By the formula (2.5.12), we have the general solution

\[ y(x) = (c_1 + c_2 \ln x)x^{\frac{1-A}{2}} = (c_1 + c_2 \ln x)x^3, \]

where \( c_1 \) and \( c_2 \) are arbitrary constants.
Case 3. Complex Roots

Suppose \((1 - A)^2 - 4B < 0\) in the equation (2.5.10). Then the equation has complex roots

\[
\lambda = \frac{1-A}{2} \pm i \sqrt{\frac{1}{4} (1-A)^2 - B}.
\]

When we let

\[
\lambda = p \pm iq, \quad p = \frac{1-A}{2}, \quad q = \sqrt{\frac{1}{4} (1-A)^2 - B},
\]

we observe \(y_1(x) = x^p \cos (q \ln x)\) and \(y_2(x) = x^p \sin (q \ln x)\) are solutions of the original equation (2.5.9) and also they are linearly independent. Therefore, the general solution of (2.5.9) is

\[
y(x) = c_1 x^p \cos (q \ln x) + c_2 x^p \sin (q \ln x),
\]

where \(c_1\) and \(c_2\) are arbitrary constants.

Example 2.5.12. Solve the differential equation

\[
y'' - \frac{1}{x} y' + \frac{4}{x^2} y = 0, \quad \text{or} \quad x^2 y'' - xy' + 4y = 0. \quad (x > 0)
\]

**Answer.** From the given equation, we deduce the equation on the power of \(x^\lambda\) like (2.5.10)

\[
\lambda^2 + (-1 - 1) \lambda + 4 = 0, \quad 0 = \lambda^2 - 2\lambda + 4, \quad \lambda = 1 \pm i\sqrt{3}.
\]

By the formula (2.5.13), we have the general solution

\[
y(x) = x^p (c_1 \cos (q \ln x) + c_2 \sin (q \ln x)) = x \left( c_1 \cos \left( \sqrt{3} \ln x \right) + c_2 \sin \left( \sqrt{3} \ln x \right) \right),
\]

where \(c_1\) and \(c_2\) are arbitrary constants.

**Remark 2.5.13 (Summary).** Solve the Euler equation

\[
y'' + \frac{A}{x} y' + \frac{B}{x^2} y = 0, \quad \text{or} \quad x^2 y'' + Axy' + By = 0 \quad (x > 0)
\]

where \(A\) and \(B\) are real numbers.

We find and solve its auxiliary equation

\[
\lambda^2 + (A - 1) \lambda + B = 0.
\]

Depending on the roots, we have three different general solutions of the Euler equation,

\[
y(x) = c_1 x^a + c_2 x^b, \quad y(x) = (c_1 + c_2 \ln x)x^a, \quad y(x) = x^p (c_1 \cos (q \ln x) + c_2 \sin (q \ln x)) .
\]

Part II. Generalization (Purely Optional)

We consider a little bit generalized Euler equation

\[
y'' + \frac{A}{ax + b} y' + \frac{B}{(ax + b)^2} y = 0, \quad x > \frac{-b}{a},
\]

where \(A, B, a\) and \(b\) are constants. If \(a = 1\) and \(b = 0\), then we have Euler’s equation which we have seen. We introduce a change of variable to solve the equation. Note that the strategy introduced in the textbook/lecture note is also based on a change of variable.
We take the change of variable

\[ u = ax + b, \quad x = \frac{u - b}{a}, \quad \frac{du}{dx} = a, \quad \frac{dx}{du} = \frac{1}{a}. \]

and so \( y(x) = y(x(u)) = (y \circ x)(u)\). Let \( Y(u) = (y \circ x)(u)\). Then by the Chain Rule,

\[
\begin{align*}
Y'(u) &= \frac{dY}{du} = \frac{d}{du} (y \circ x)(u) = \frac{d}{dx} (y \circ x)(u) \frac{dx}{du} = \frac{dy}{dx} \frac{dx}{du} = y'(x) \frac{dx}{du} = y'(x) \frac{1}{a}, \\
Y''(u) &= \frac{dY'}{du} = \frac{dY}{d(1/a'y(x))} \left( \frac{1}{a'} \frac{dy}{dx} \right) = \frac{1}{a} \frac{d}{du} \left( y'(x) \right) = \frac{1}{a} \frac{d}{dx} \left( y'(x) \right) \frac{dx}{du} = \frac{1}{a^2} y''(x). 
\end{align*}
\]

Thus we have \( y''(x) = a^2 Y''(u) \) and \( y'(x) = aY'(u) \) and so the Euler equation for \( y(x) \) becomes

\[
a^2 Y''(u) + \frac{A}{u} aY'(u) + \frac{B}{u^2} Y(u) = 0, \quad \text{i.e.,} \quad Y''(u) + \frac{A/a}{u} Y'(u) + \frac{B/a^2}{u^2} Y(u) = 0.
\]

The resulting differential equation has the characteristic equation

\[
\lambda^2 + \left( \frac{A}{a} - 1 \right) \lambda + \frac{B}{a^2} = 0, \quad \text{i.e.,} \quad a^2 \lambda^2 + a(A-a) \lambda + B = 0,
\]

which has the roots

\[
\lambda = -a(A-a) \pm \sqrt{a^2(A-a)^2 - 4a^2B} = -a + \frac{A-a}{2a} \pm \frac{1}{2a} \sqrt{(A-a)^2 - 4B}.
\]

Just like other strategies, we have three cases: two distinct real roots, repeated root, and two complex roots.

**Case 1. Two Distinct Real Roots:** Suppose the characteristic equation has two distinct real roots \( \lambda = \alpha \) and \( \lambda = \beta \). Then we have two solutions

\[
Y_1(u) = u^\alpha, \quad Y_2(u) = u^\beta
\]

and the general solution is

\[
Y(u) = c_1 Y_1(u) + c_2 Y_2(u) = c_1 u^\alpha + c_2 u^\beta,
\]

where \( c_1 \) and \( c_2 \) are arbitrary constants. Therefore, the general solution \( y(x) \) of the original Euler equation is

\[
y(x) = c_1 (ax + b)^\alpha + c_2 (ax + b)^\beta.
\]

**Example 2.5.14.** Solve the differential equation

\[
(1 + 2x)^2 y'' + 8(1 + 2x)y' - 16y = 0, \quad x > -\frac{1}{2}.
\]

**ANSWER.** We use the change of variable \( u = 2x + 1 \) and let \( Y(u) = y(x) \). Then we have

\[
y'(x) = Y'(u) \frac{du}{dx} = 2Y'(u), \quad y''(x) = 4Y''(u)
\]

and the equation is transformed

\[
4u^2 Y''(u) + 16u Y'(u) - 16Y(u) = 0, \quad \text{i.e.,} \quad Y'' + \frac{4}{u} Y' - \frac{4}{u^2} Y = 0.
\]

It has the characteristic equation

\[
0 = \lambda^2 + 3\lambda - 4 = (\lambda + 4)(\lambda - 1), \quad \lambda = -4, \lambda = 1.
\]

Thus, we have two solutions \( Y_1(u) = u^{-4} \) and \( Y_2(u) = u \) and the general solution

\[
Y(u) = c_1 Y_1(u) + c_2 Y_2(u) = c_1 u^{-4} + c_2 u.
\]

Therefore, finally the general solution of the original differential equation is obtained by

\[
y(x) = c_1 (2x + 1)^{-4} + c_2 (2x + 1).
\]

For the Case 2. Repeated Root and Case 3. Complex Roots, we can argue similarly as we did for other strategies.
§2.6 Nonhomogeneous Equation $y'' + p(x)y' + q(x)y = f(x)$.

Suppose that we can find the general solution $y_h$ of the linear homogeneous equation

$$y'' + p(x)y' + q(x)y = 0.$$  \hspace{1cm} (2.6.1)

Then the linear nonhomogeneous equation

$$y'' + p(x)y' + q(x)y = f(x)$$  \hspace{1cm} (2.6.2)

has the general solution $y = y_h + y_p$, where $y_p$ is any particular solution of the nonhomogeneous equation (2.6.2). That is, the strategy for solving (2.6.2) is as follows:

Step 1. Find the general solution $y_h$ of its homogeneous equation (2.6.1).

Step 2. Find a particular solution $y_p$ of the nonhomogeneous equation (2.6.2).

Step 3. We conclude the general solution of the equation (2.6.2) $y = y_h + y_p$.

In Sections 2.3, 2.4 and 2.5, we have studied on how to find the general solution of a homogeneous equation. So the work on the Step 1 can be done by the strategies in those previous sections. In this section, we mainly focus on the Step 2, i.e., how to find a particular solution $y_p$ of a nonhomogeneous equation (2.6.2). We will discuss two methods: Method of Variation of Parameters and Method of Undetermined Coefficients.

\section*{2.6.1 Method of Variation of Parameters.}

We recall that if $y_1$ and $y_2$ forms the fundamental set of solutions of the homogeneous equation (2.6.1), then the general solution of (2.6.1) is given by

$$y_h(x) = c_1y_1(x) + c_2y_2(x),$$  \hspace{1cm} (2.6.3)

where $c_1$ and $c_2$ are arbitrary constants.

We look for a particular solution $y_p$ of the equation (2.6.2) in the form

$$y_p(x) = u(x)y_1(x) + v(x)y_2(x),$$  \hspace{1cm} (2.6.4)

where $u(x)$ and $v(x)$ will be determined and it is formed by replacing constants $c_1$ and $c_2$ in (2.6.3) with functions $u(x)$ and $v(x)$. This technique is called the method of variation of parameters.

Let us find $u(x)$ and $v(x)$ explicitly. Since $y_p$ in (2.6.4) should be a particular solution of (2.6.2), it should satisfy

\begin{align*}
    f(x) &= y_p'' + p(x)y_p' + q(x)y_p = (uy_1 + vy_2)'' + p(uy_1 + vy_2)' + q(uy_1 + vy_2) \\
    &= (u'y_1 + v'y_2)' + (uy_1' + vy_2')' + p(u'y_1 + v'y_2) + q(uy_1 + vy_2) \\
    &= (u'y_1 + v'y_2)' + (u'y_1' + vy_2') + p(u'y_1 + v'y_2) + q(uy_1 + vy_2) \\
    &= u(y_1'' + py_1' + qy_1) + v(y_2'' + py_2' + qy_2) + (u'y_1 + v'y_2)' + (u'y_1' + vy_2') + p(u'y_1 + v'y_2) \\
    &= u(0) + v(0) + (u'y_1 + v'y_2)' + (u'y_1' + vy_2') + p(u'y_1 + v'y_2) \\
    &= (u'y_1 + v'y_2)' + (u'y_1' + vy_2') + p(u'y_1 + v'y_2),
\end{align*}

where $y_1'' + py_1' + qy_1 = 0$ and $y_2'' + py_2' + qy_2 = 0$, because $y_1$ and $y_2$ are solutions of (2.6.1). Suppose we give a condition on $u$ and $v$ as follows

$$u'y_1 + v'y_2 = 0.$$

Then the equation above is simplified by

$$f(x) = (u'y_1 + v'y_2)' + (u'y_1' + v'y_2') + p(u'y_1 + v'y_2).$$
\[(0)' + (u'y_1 + v'y_2) + p(0) = u'y_1' + v'y_2'.\]

Therefore, we can deduce two equations for \(u\) and \(v\):

\[u'y_1 + v'y_2 = 0 \quad \text{and} \quad u'y_1' + v'y_2' = f(x).\]

Since we have two equations and two unknowns \(u'\) and \(v'\), we can solve for \(u'\) and \(v'\), in fact,

\[
u' = -\frac{y_2f}{W}, \quad v' = \frac{y_1f}{W}, \quad u = -\int \frac{y_2f}{W} \, dx, \quad v = \int \frac{y_1f}{W} \, dx, \tag{2.6.5}
\]

where \(W(x)\) is the Wronskian of \(y_1\) and \(y_2\), i.e., \(W(x) = y_1(x)y_2'(x) - y_1'(x)y_2(x)\). (Since \(y_1\) and \(y_2\) forms the fundamental set of solutions, we should have \(W(x) \neq 0\) for all \(x\).) Finally, we obtain a particular solution \(y_p(x)\) of the nonhomogeneous equation (2.6.2),

\[y_p(x) = u(x)y_1(x) + v(x)y_2(x) = \left[-\int \frac{y_2(x)f(x)}{W(x)} \, dx\right] y_1(x) + \left[\int \frac{y_1(x)f(x)}{W(x)} \, dx\right] y_2(x). \tag{2.6.6}
\]

The general solution of the nonhomogeneous equation (2.6.2) is eventually obtained by

\[y(x) = y_h(x) + y_p(x) = c_1y_1(x) + c_2y_2(x) + \left[-\int \frac{y_2(x)f(x)}{W(x)} \, dx\right] y_1(x) + \left[\int \frac{y_1(x)f(x)}{W(x)} \, dx\right] y_2(x) = \left[c_1 - \int \frac{y_2(x)f(x)}{W(x)} \, dx\right] y_1(x) + \left[c_2 + \int \frac{y_1(x)f(x)}{W(x)} \, dx\right] y_2(x).\]

**Remark 2.6.1.**

1. Suppose \(f(x) = 0\). Then the nonhomogeneous equation (2.6.2) becomes the homogeneous equation (2.6.1) and the particular solution \(y_p(x)\) of the nonhomogeneous equation in (2.6.6) becomes

\[y_p(x) = \left[-\int \frac{y_2(x)0}{W(x)} \, dx\right] y_1(x) + \left[\int \frac{y_1(x)0}{W(x)} \, dx\right] y_2(x) = \left[-\int 0 \, dx\right] y_1(x) + \left[\int 0 \, dx\right] y_2(x) = c_1y_1(x) + c_2y_2(x) = y_h(x),\]

which is the general solution of the homogeneous equation.

2. Please do not try to memorize all of them. In the method of variation of parameters, the results (2.6.5) are worthwhile to be memorized.

3. One should not be confused. In the formula (2.6.6), the coefficient of \(y_1(x)\) has \(y_2(x)\) in the integral, while the coefficient of \(y_2(x)\) has \(y_1(x)\) in the integral. When we have two solutions, it is up to us to choose which one is \(y_1\) and which one is \(y_2\). But, once we fixed them, they should be carefully plugged into the formula (2.6.6).

We summarize the steps of the method.

**Theorem 2.6.2 (Method of Variation of Parameters).** A particular solution of \(y'' + py' + qy = f\) can be found through the following steps.

**Step 1. General Solution.** We find the general solution \(y_h = c_1y_1 + c_2y_2\) of the equation \(y'' + py' + qy = 0\).

**Step 2. Method of Variation of Parameters.** We set \(y_p = uy_1 + vy_2\) and find \(u\) and \(v\) satisfying

\[u'y_1 + v'y_2 = 0 \quad \text{and} \quad u'y_1' + v'y_2' = f.\]
In fact, those \( u \) and \( v \) are obtained by
\[
u(x) = - \int \frac{y_2(x)f(x)}{W(x)} \, dx,
\]
\[
u(x) = \int \frac{y_1(x)f(x)}{W(x)} \, dx,
\]
where \( W(x) \) is the Wronskian of \( y_1 \) and \( y_2 \). Hence, the particular solution \( y_p \) is obtained by
\[
y_p = uy_1 + vy_2 = \left( - \int \frac{y_2(x)f(x)}{W(x)} \, dx \right) y_1 + \left( \int \frac{y_1(x)f(x)}{W(x)} \, dx \right) y_2.
\]
Now let us see how to use the results above through the examples.

**Example 2.6.3.** A differential equation is given by
\[y'' + 4y = \sec x, \quad -\frac{\pi}{4} < x < \frac{\pi}{4}.
\]

**Step 1.** \( y_h \). Find two solutions \( y_1 \) and \( y_2 \) and the general solution \( y_h(x) = c_1y_1(x) + y_2(x) \) of its homogeneous equation \( y'' + 4y = 0, \ -\frac{\pi}{4} < x < \frac{\pi}{4} \).

**Step 2.** \( y_p \). Find a particular solution \( y_p = uy_1 + vy_2 \) of the given nonhomogeneous equation.

**Step 3.** \( y = y_h + y_p \). Find the general solution \( y = y_h + y_p \) of the given nonhomogeneous equation.

**Answer.** **Step 1.** \( y_h \). The homogeneous equation has the characteristic equation
\[\lambda^2 + 4 = 0, \quad \lambda = \pm 2i.
\]
Thus, we have two solutions
\[y_1(x) = e^{0x} \cos (2x) = \cos (2x), \quad \text{and} \quad y_2(x) = e^{0x} \sin (2x) = \sin (2x),
\]
and the general solution of the homogeneous equation is
\[y_h(x) = c_1y_1(x) + c_2y_2(x) = c_1 \cos (2x) + c_2 \sin (2x).
\]

**Step 2.** \( y_p \). We find a particular solution \( y_p(x) = u(x)y_1(x) + v(x)y_2(x) \), precisely, \( u(x) \) and \( v(x) \):
\[y_p(x) = u(x)\cos (2x) + v(x)\sin (2x),
\]
by using the formula (2.6.6). The Wronskian of \( y_1(x) = \cos (2x) \) and \( y_2(x) = \sin (2x) \) is found by
\[W(x) = y_1'y_2' - y_1y_2' = 2\cos (2x)\cos (2x) + 2\sin (2x)\sin (2x) = 2(\cos^2 (2x) + \sin^2 (2x)) = 2.
\]
The formula (2.6.6) implies
\[
u'(x) = -\frac{1}{2} \sin (2x) \sec x = -\frac{1}{2} 2\sin x \cos x \sec x = -\sin x,
\]
\[
u'(x) = \frac{1}{2} \cos (2x) \sec x = \frac{1}{2} (2\cos^2 x - 1) \sec x = \cos x - \frac{1}{2} \sec x.
\]
Integrating them, we get
\[
u(x) = -\int \sin x \, dx = \cos x, \quad v(x) = \int \left( \cos x - \frac{1}{2} \sec x \right) \, dx = \sin x - \frac{1}{2} \ln |\sec x + \tan x|.
\]
(We do not have to put the constants of integration, because this is the work for finding a particular solution \( y_p \). We can take the constants to be zero.) Therefore, we deduce a particular solution of the nonhomogeneous equation
\[y_p(x) = u(x)\cos (2x) + v(x)\sin (2x) = \cos x \cos (2x) + \left( \sin x - \frac{1}{2} \ln |\sec x + \tan x| \right) \sin (2x).
\]

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**Step 3.** $y = y_h + y_p$. Since we have the general solution $y_h$ in (2.6.7) of the homogeneous equation and a particular solution $y_p$ in (2.6.8) of the nonhomogeneous equation, therefore, the general solution of the nonhomogeneous equation is given by

$$y(x) = y_h(x) + y_p(x)$$

$$= c_1 \cos (2x) + c_2 \sin (2x) + \cos x \cos (2x) + \left( \sin x - \frac{1}{2} \ln |\sec x + \tan x| \right) \sin (2x)$$

$$= (c_1 + \cos x) \cos (2x) + \left( c_2 + \sin x - \frac{1}{2} \ln |\sec x + \tan x| \right) \sin (2x),$$

where $c_1$ and $c_2$ are arbitrary constants.

**Example 2.6.4.** A differential equation is given by

$$y'' - \frac{4}{x} y' + \frac{4}{x^2} y = x^2 + 1, \quad (x > 0).$$

**Step 1.** $y_h$. Find two solutions $y_1$ and $y_2$ and the general solution $y_h(x) = c_1 y_1(x) + y_2(x)$ of its homogeneous equation $y'' - (4/x) y' + (4/x^2) y = 0$, $x > 0$.

**Step 2.** $y_p$. Find a particular solution $y_p = uy_1 + vy_2$ of the given nonhomogeneous equation.

**Step 3.** $y = y_h + y_p$. Find the general solution $y = y_h + y_p$ of the given nonhomogeneous equation.

**Answer.** **Step 1.** $y_h$. The homogeneous equation has the auxiliary equation

$$0 = \lambda^2 + (-4 - 1) \lambda + 4 = \lambda^2 - 5\lambda + 4 = (\lambda - 1)(\lambda - 4), \quad \lambda = 1, \lambda = 4.$$

Thus, we have two solutions

$$y_1(x) = x, \quad \text{and} \quad y_2(x) = x^4,$$

and the general solution of the homogeneous equation is

$$y_h(x) = c_1 y_1(x) + c_2 y_2(x) = c_1 x + c_2 x^4. \quad (2.6.9)$$

(For the method used in Step 1, please read the Appendix 2.5.1 in the lecture note on Section 2.5 Euler’s Equation.)

**Step 2.** $y_p$. We find a particular solution $y_p(x) = u(x)y_1(x) + v(x)y_2(x)$, precisely, $u(x)$ and $v(x)$:

$$y_p(x) = u(x)x + v(x)x^4,$$

by using the formula (2.6.6). The Wronskian of $y_1(x) = x$ and $y_2(x) = x^4$ is found by

$$W(x) = y_1'y_2 - y_1y_2' = x(4x^3) - 1 (x^4) = 3x^4,$$

which is not zero for $x > 0$. The formula (2.6.6) implies

$$u'(x) = -\frac{x^4(x^2 + 1)}{3x^4} = -\frac{1}{3}(x^2 + 1), \quad v'(x) = \frac{x(x^2 + 1)}{3x^4} = \frac{1}{3} \left( \frac{1}{x} + \frac{1}{x^3} \right).$$

Integrating them, we get

$$u(x) = -\frac{1}{3} \int (x^2 + 1) \, dx = -\frac{1}{3} \left( \frac{1}{3} x^3 + x \right), \quad v(x) = \frac{1}{3} \int \left( \frac{1}{x} + \frac{1}{x^3} \right) \, dx = \frac{1}{3} \left( \ln x - \frac{1}{2x^2} \right).$$

(We do not have to put the constants of integration, because this is the work for finding a particular solution $y_p$. We can take the constants to be zero.) Therefore, we deduce a particular solution of the nonhomogeneous equation

$$y_p(x) = u(x)x + v(x)x^4 = -\frac{1}{3} \left( \frac{1}{3} x^3 + x \right) x + \frac{1}{3} \left( \ln x - \frac{1}{2x^2} \right) x^4 = -\frac{1}{9} x^4 - \frac{1}{2} x^2 + \frac{1}{3} x^4 \ln x. \quad (2.6.10)$$
Step 3. $y = y_h + y_p$. Since we have the general solution $y_h$ in (2.6.9) of the homogeneous equation and a particular solution $y_p$ in (2.6.10) of the nonhomogeneous equation, therefore, the general solution of the nonhomogeneous equation is given by

$$y(x) = y_h(x) + y_p(x) = c_1 x + c_2 x^4 - \frac{1}{9} x^4 - \frac{1}{2} x^2 + \frac{1}{3} x^4 \ln x$$

$$= c_1 x + \left( c_2 - \frac{1}{9} \right) x^4 - \frac{1}{2} x^2 + \frac{1}{3} x^4 \ln x, \quad (x > 0)$$

where $c_1$ and $c_2$ are arbitrary constants.

□

2.6.2 Method of Undetermined Coefficients.

The method in this topic is only applicable for constant coefficient nonhomogeneous equation

$$y'' + Ay' + By = f(x), \quad (2.6.11)$$

where $A$ and $B$ are constants. The goal is to find a particular solution $y_p$ of (2.6.11) without finding the solution $y_h$ of its homogeneous part.

Theorem 2.6.5 (Method of Undetermined Coefficients).

Step 1. From $f(x)$, make a first conjecture for the form of $y_p$. (How? See the table below.)

Step 2. Solve its homogeneous equation $y'' + Ay' + By = 0$.

1. If a solution of the homogeneous equation appears in any term of the conjectured form for $y_p$, then modify this form by multiplying it by $x$.
2. If the modified function still occurs in a solution of the homogenous equation, then multiply it by $x$ again (so the original $y_p$ is multiplied by $x^2$ in this case).

Step 3. Substitute the final proposed $y_p$ into the nonhomogeneous equation $y'' + Ay' + By = f(x)$ and solve for its coefficients.

Here is the list of functions which we may try in the Step 1 of formulating $y_p$. $P(x)$, $Q(x)$ and $R(x)$ represent polynomials of the same degree. (Please memorize the following table.)

<table>
<thead>
<tr>
<th>$f(x)$</th>
<th>Initial Guess for $y_p$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P(x)$</td>
<td>$Q(x)$</td>
</tr>
<tr>
<td>$ce^{ax}$</td>
<td>$de^{ax}$</td>
</tr>
<tr>
<td>$\alpha \cos(bx)$ or $\beta \sin(bx)$</td>
<td>$c \cos(bx) + d \sin(bx)$</td>
</tr>
<tr>
<td>$P(x)e^{ax}$</td>
<td>$Q(x)e^{ax}$</td>
</tr>
<tr>
<td>$P(x)\cos(bx)$ or $P(x)\sin(bx)$</td>
<td>$Q(x)\cos(bx) + R(x)\sin(bx)$</td>
</tr>
<tr>
<td>$P(x)e^{ax} \cos(bx)$ or $P(x)e^{ax} \sin(bx)$</td>
<td>$Q(x)e^{ax} \cos(bx) + R(x)e^{ax} \sin(bx)$</td>
</tr>
</tbody>
</table>

Let us see how the method above works through examples.

Example 2.6.6 (Polynomial $f$). Use the method of undetermined coefficients to find a particular solution of the constant coefficient nonhomogeneous equation

$$y'' - 4y = 8x^2 - 2x. \quad (2.6.12)$$
ANSWER. **Step 1.** Since \( f(x) = 8x^2 - 2x \) is a polynomial of degree 2, we guess that a particular solution \( y_p \) may be a polynomial of degree 2, and so we set

\[
y_p(x) = ax^2 + bx + c,
\]

where \( a, b \) and \( c \) are constants to be determined later.

**Step 2.** The homogeneous equation

\[
y'' - 4y = 0
\]

has two solutions \( y_1(x) = e^{2x} \) and \( y_2(x) = e^{-2x} \) and its general solution \( y_h(x) = c_1 e^{2x} + c_2 e^{-2x} \). (Why? See the Section 2.4.) Since any solution \( y_1 \) or \( y_2 \) of the homogeneous equation does not appear in the conjectured form (2.6.13) for \( y_p \), so we move to next step.

**Step 3.** Putting the guess (2.6.13) into the given equation (2.6.12), we have

\[
8x^2 - 2x = y_p'' - 4y_p = 2a - 4ax^2 - 4bx - 4c, \quad 4(a + 2)x^2 + 2(2b - 1)x + 2(2c - a) = 0, \quad \text{for all } x.
\]

In general,

\[
a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0 = 0 \quad \text{for all } x
\]

if and only if \( a_n = a_{n-1} = \cdots = a_2 = a_1 = a_0 \).

So we should get

\[
\begin{align*}
a + 2 &= 0, \\
2b - 1 &= 0, \\
2c - a &= 0,
\end{align*}
\]

i.e., \( a = -2, \ b = 1/2 \) and \( c = -1 \) and the particular solution is

\[
y_p(x) = -2x^2 + \frac{1}{2} x^2 - 1.
\]

One can verify by substituting it into the given nonhomogeneous equation.

**General Solution of (2.6.12).** The general solution \( y(x) = y_h(x) + y_p(x) \) of the nonhomogeneous equation (2.6.12) is given by

\[
y(x) = y_h(x) + y_p(x) = c_1 e^{2x} + c_2 e^{-2x} - 2x^2 + \frac{1}{2} x - 1. \quad \Box
\]

**Example 2.6.7 (Exponential Function f).** Use the method of undetermined coefficients to find a particular solution of the constant coefficient nonhomogeneous equation

\[
y'' + 2y' - 3y = 4e^{2x}. \quad (2.6.14)
\]

ANSWER. **Step 1.** Since \( f(x) = 4e^{2x} \) is an exponential function, we guess that a particular solution \( y_p \) may be an exponential function, and so we set

\[
y_p(x) = ae^{2x}, \quad (2.6.15)
\]

where \( a \) is a constant to be determined later.

**Step 2.** The homogeneous equation

\[
y'' + 2y' - 3y = 0
\]

has two solutions \( y_1(x) = e^{-3x} \) and \( y_2(x) = e^x \) and its general solution \( y_h(x) = c_1 e^{-3x} + c_2 e^x \). (Why? See the Section 2.4.) Since any solution \( y_1 \) or \( y_2 \) of the homogeneous equation does not appear in the conjectured form (2.6.15) for \( y_p \), so we move to next step.

**Step 3.** Putting the guess (2.6.15) into the given equation (2.6.14), we have

\[
4e^{2x} = y_p'' + 2y_p' - 3y_p = 4ae^{2x} + 4ae^{2x} - 3ae^{2x} = 5ae^{2x}, \quad \text{i.e.,} \quad e^{2x} (5a - 4) = 0.
\]
An exponential function can never be zero. So we should get
\[ 5a - 4 = 0, \quad a = \frac{4}{5}, \]
and the particular solution is
\[ y_p(x) = \frac{4}{5} e^{2x}. \]
One can verify by substituting it into the given nonhomogeneous equation.

**General Solution of (2.6.14).** The general solution \( y(x) = y_h(x) + y_p(x) \) of the nonhomogeneous equation (2.6.14) is given by
\[ y(x) = y_h(x) + y_p(x) = c_1 e^{-3x} + c_2 e^x + \frac{4}{5} e^{2x}. \]

**Example 2.6.8 (Trigonometric Function f).** Use the method of undetermined coefficients to find a particular solution of the constant coefficient nonhomogeneous equation
\[ y'' - 5y' + 6y = -3 \sin(2x). \quad (2.6.16) \]

**Answer. Step 1.** Since \( f(x) = -3 \sin(2x) \) is a trigonometric function, we guess that a particular solution \( y_p \) may be a combination of trigonometric functions, and so we set
\[ y_p(x) = a \cos(2x) + b \sin(2x), \quad (2.6.17) \]
where \( a \) and \( b \) are constants to be determined later.

**Step 2.** The homogeneous equation
\[ y'' - 5y' + 6y = 0 \]
has two solutions \( y_1(x) = e^{3x} \) and \( y_2(x) = e^{2x} \) and its general solution \( y_h(x) = c_1 e^{3x} + c_2 e^{2x} \). (Why? See the Section 2.4.) Since any solution \( y_1 \) or \( y_2 \) of the homogeneous equation does not appear in the conjectured form (2.6.17) for \( y_p \), so we move to next step.

**Step 3.** Putting the guess (2.6.17) into the given equation (2.6.16), we have
\[
-3 \sin(2x) = y''_p - 5y'_p + 6y_p
= -4a \cos(2x) - 4b \sin(2x) - 5 (-2a \sin(2x) + 2b \cos(2x)) + 6 (a \cos(2x) + b \sin(2x))
\]
i.e.,
\[ 0 = 2 (a - 5b) \cos(2x) + (10a + 2b + 3) \sin(2x), \quad \text{for all} \ x. \]
In general,
\[ A \cos x + B \sin x = 0 \quad \text{for all} \ x \quad \text{if and only if} \quad A = 0 = B. \]

In **linear algebra** or **maybe calculus 4**, we have the following theorem:

**Suppose two functions** \( p(x) \) **and** \( q(x) \) **are linearly independent. Then,**
\[ Ap(x) + Bq(x) = 0 \quad \text{for all} \ x \quad \text{if and only if} \quad A = 0 = B. \]
Since \( \cos(2x) \) and \( \sin(2x) \) are linearly independent (the Wronskian Test), so we should get
\[ a - 5b = 0 \quad \text{and} \quad 10a + 2b + 3 = 0. \]
Solving the equations for \( a \) and \( b \), we get \( a = -15/52 \) and \( b = -3/52 \) and the particular solution is
\[ y_p(x) = -\frac{15}{52} \cos(2x) - \frac{3}{52} \sin(2x). \]
One can verify by substituting it into the given nonhomogeneous equation.

**General Solution of (2.6.16).** The general solution \( y(x) = y_h(x) + y_p(x) \) of the nonhomogeneous equation (2.6.16) is given by
\[ y(x) = y_h(x) + y_p(x) = c_1 e^{3x} + c_2 e^{2x} - \frac{15}{52} \cos(2x) - \frac{3}{52} \sin(2x). \]
There are problems that the main ideas in the *method of undetermined coefficients* may not work, however, still we can solve the problem with a slight modification of the method. Let us see an example.

**Example 2.6.9.** Use the *method of undetermined coefficients* to find a particular solution of the differential equation

\[ y'' + 2y' - 3y = 8e^x. \]  

**Answer.** **Step 1.** Since \( f(x) = 8e^x \) is an exponential function, we guess that a particular solution \( y_p \) may be an exponential function, and so we set

\[ y_p(x) = ae^x, \]  

where \( a \) is a constant to be determined later.

**Step 2.** The homogeneous equation

\[ y'' + 2y' - 3y = 0 \]

has two solutions \( y_1(x) = e^{-3x} \) and \( y_2(x) = e^x \) and its general solution \( y_h(x) = c_1e^{-3x} + c_2e^x \). (Why? See the Section 2.4.) Since the solution \( y_2 = e^x \) of the homogeneous equation does appear in the conjectured form (2.6.19) for \( y_p \), so we modify our guess by multiplying by \( x \) and try a new guess

\[ y_p(x) = axe^x. \]  

**Step 3.** Putting the new guess (2.6.20) into the given equation (2.6.18), we have

\[ 8e^x = y_p'' + 2y_p' - 3y_p = 2ae^x + axe^x + 2(2ae^x) - 3ae^x = 4ae^x, \]

\[ 8e^x = 4ae^x, \quad \text{i.e.,} \quad a = 2. \]

Hence, we deduce a particular solution \( y_p(x) = 2xe^x \). One can verify by substituting it into the given nonhomogeneous equation.

**General Solution of (2.6.18).** The general solution \( y(x) = y_h(x) + y_p(x) \) of the nonhomogeneous equation (2.6.18) is given by

\[ y(x) = y_h(x) + y_p(x) = c_1e^{-3x} + c_2e^x + 2xe^x. \]  

**Exercise 2.6.10.** Use the *method of undetermined coefficients* to find a particular solution of the differential equation

\[ 2y'' - y' + y = x + e^x. \]  

(Final Exam of Fall 2009)

If we use the *method of undetermined coefficients* to find a particular solution \( y_p \) and eventually the general solution \( y = y_h + y_p \), then it is suggested to follow the steps in the theorem.

**Theorem 2.6.11 (General Solution of Nonhomogeneous Equation).**

**Step 1.** General Solution \( y_h \) of Homogeneous Equation

**Step 2.** Particular Solution \( y_p \)

**Step 3.** General Solution \( y = y_h + y_p \) of Nonhomogeneous Equation

**Example 2.6.12.** Solve the nonhomogeneous equation

\[ y'' - 6y' + 9y = 5e^{3x}. \]

**Answer.** **Step 1.** General Solution \( y_h \). To avoid a useless guess on a particular solution, we first find the general solution \( y_h \) of its homogeneous equation

\[ y'' - 6y' + 9y = 0. \]

It has the characteristic equation

\[ 0 = \lambda^2 - 6\lambda + 9 = (\lambda - 3)^2, \]

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which has repeated root $\lambda = 3$. Hence, the general solution $y_h$ is obtained by

$$y_h(x) = (c_1 + c_2x)e^{3x}.$$  

**Step 2. Particular Solution** $y_p$. Since the general solution $y_h$ has terms $e^{3x}$ and $xe^{3x}$, so we try a particular solution $y_p(x) = ax^2e^{3x}$, where $e^{3x}$ in $y_p$ comes from the function $f(x) = 5e^{3x}$. Putting $y_p(x) = ax^2e^{3x}$ into the nonhomogeneous equation, we have

$$5e^{3x} = (ax^2e^{3x})'' - 6(ax^2e^{3x})' + 9y$$

$$= 2ae^{3x} + 12ae^x + 9a^2e^{3x} - 6(2axe^{3x} + 3ax^2e^{3x}) + 9ax^2e^{3x} = 2ae^{3x}, \quad i.e., \quad a = \frac{5}{2}.$$  

Thus, we have a particular solution $y_p(x) = \frac{5}{2}x^2e^{3x}$.

**Step 3. General Solution.** From Step 1 and Step 2, we conclude the general solution of the nonhomogeneous equation is

$$y(x) = y_h(x) + y_p(x) = (c_1 + c_2x)e^{3x} + \frac{5}{2}x^2e^{3x} = \left(c_1 + c_2x + \frac{5}{2}x^2\right)e^{3x}. \quad \square$$

**Example 2.6.13.** Solve the nonhomogeneous equation

$$y'' + 9y = -4x\sin(3x).$$

**Answer.** **Step 1. General Solution** $y_h$. To avoid a useless guess on a particular solution, we first find the general solution $y_h$ of its homogeneous equation

$$y'' + 9y = 0.$$  

It has the characteristic equation

$$\lambda^2 + 9 = 0,$$

which has two complex roots $\lambda = \pm 3i$. Hence, the general solution $y_h$ is obtained by

$$y_h(x) = c_1 \cos(3x) + c_2 \sin(3x).$$

**Step 2. Particular Solution** $y_p$. For the function $f(x) = -4x\sin(3x)$, we follow the suggested form in the table above. So we try a particular solution

$$y_p(x) = (ax + b)\cos(3x) + (cx + d)\sin(3x).$$

But unfortunately, it does not work. So we modify by multiplying it by $x$, i.e., we try

$$y_p(x) = x(ax + b)\cos(3x) + x(cx + d)\sin(3x) = (ax^2 + bx)\cos(3x) + (cx^2 + dx)\sin(3x).$$

Putting it into the nonhomogeneous equation, we have

$$-4x\sin(3x) = \left[(ax^2 + bx)\cos(3x) + (cx^2 + dx)\sin(3x)\right]'$$

$$+ 9 \left[(ax^2 + bx)\cos(3x) + (cx^2 + dx)\sin(3x)\right],$$

$$i.e., \quad 0 = (2a + 6d)\cos(3x) + (-6b + 2c)\sin(3x) + 12cx\cos(3x) + (-12a + 4)x\sin(3x),$$

for all $x$. In **Linear Algebra** or **Calculus 4 on Basis**, we have the following theorem:

Suppose functions $p(x), q(x), r(x)$ and $s(x)$ are linearly independent. Then, $Ap(x) + Bq(x) + Cr(x) + Ds(x) = 0$ for all $x$ if and only if $A = 0, B = 0, C = 0$ and $D = 0$. 


Since \( \cos(3x), \sin(3x), x\cos(3x) \) and \( x\sin(3x) \) are linearly independent (Wronskian Test), we should get

\[
2a + 6d = 0, \quad \text{and} \quad -6d + 2c = 0, \quad \text{and} \quad 12c = 0, \quad \text{and} \quad -12a + 4 = 0.
\]

Solving them for \( a, b, c \) and \( d \), we have

\[
a = \frac{1}{3}, \quad b = 0 = c, \quad d = \frac{1}{9}.
\]

Thus, we have a particular solution

\[
y_p(x) = \frac{1}{3}x^2 \cos(3x) - \frac{1}{9} x \sin(3x).
\]

**Step 3. General Solution.** From Step 1 and Step 2, we conclude the general solution of the nonhomogeneous equation is

\[
y(x) = y_h(x) + y_p(x) = c_1 \cos(3x) + c_2 \sin(3x) + \frac{1}{3} x^2 \cos(3x) - \frac{1}{9} x \sin(3x)
\]

\[
= \left( c_1 + \frac{1}{3} x^2 \right) \cos(3x) + \left( c_2 - \frac{1}{9} x \right) \sin(3x).
\]

There are certain nonconstant coefficient nonhomogeneous equation which can be transformed constant coefficient nonhomogeneous equation. For this case, we may use the method of undetermined coefficients on the transformed constant coefficient equation and transform the solution back to the original equation. See the example below.

**Example 2.6.14.** Solve the nonconstant coefficient nonhomogeneous differential equation

\[
y'' - \frac{5}{x} y' + \frac{8}{x^2} y = \frac{2 \ln x}{x^2} \quad \text{or} \quad x^2 y'' - 5xy' + 8y = 2 \ln x, \quad (x > 0)
\]

of which homogeneous equation is Euler’s equation.

**Answer.** **Step 1. Transform to Constant Coefficient Equation.** Since the given equation has Euler’s equation as its homogeneous equation. This observation makes us to use the change of variable \( x = e^t \) or equivalently \( t = \ln x \). The change of variable implies that the given nonhomogeneous equation turns to be

\[
Y(t) - 6Y'(t) + 8Y(t) = 2t.
\]

(How to convert? See the Section 2.5 Euler’s Equation.) Since the transformed equation has constant coefficients, we use the method of undetermined coefficients.

**Step 2. General Solution \( Y_h \).** From the Section 2.5 Euler’s Equation, the homogeneous equation

\[
Y(t) - 6Y'(t) + 8Y(t) = 0
\]

has the general solution \( Y_h(t) = c_1 e^{2t} + c_2 e^{4t} \).

**Step 3. Particular Solution \( Y_p \).** By the method of undetermined coefficients, the transformed equation has the particular solution \( Y_p(t) = \frac{1}{4} t + \frac{3}{16} \).

**Step 4. General Solution.** From Steps above, we conclude the transformed equation has the general solution

\[
Y(t) = Y_h(t) + Y_p(t) = c_1 e^{2t} + c_2 e^{4t} + \frac{1}{4} t + \frac{3}{16}.
\]

Our purpose is the general solution of the original equation. Thus we use the change of variable again to deduce the general solution of the original equation

\[
y(x) = c_1 x^2 + c_2 x^4 + \frac{1}{4} \ln x + \frac{3}{16}. \quad \square
\]
2.6.3 Principle of Superposition.
We consider the equation
\[ y'' + p(x)y' + q(x)y = f_1(x) + f_2(x) + \ldots + f_N(x). \] (2.6.21)

Suppose \( y_{p_i} \) is a solution of the equation
\[ y'' + p(x)y' + q(x)y = f_i(x), \]
where \( i = 1, 2, \ldots, N \). Then, \( y_{p_1} + y_{p_2} + \ldots + y_{p_N} \) is a solution of the equation (2.6.21). This is called the Principle of Superposition.

PROOF. We put \( y_{p_1} + y_{p_2} + \ldots + y_{p_N} \) into the equation (2.6.21).
\[
\left( y_{p_1} + y_{p_2} + \ldots + y_{p_N} \right)'' + p(x) \left( y_{p_1} + y_{p_2} + \ldots + y_{p_N} \right)' + q \left( y_{p_1} + y_{p_2} + \ldots + y_{p_N} \right)
= \left( y_{p_1}'' + py_{p_1}' + qy_{p_1} \right) + \left( y_{p_2}'' + py_{p_2}' + qy_{p_2} \right) + \ldots + \left( y_{p_N}'' + py_{p_N}' + qy_{p_N} \right)
= f_1 + f_2 + \ldots + f_N.
\]

Thus, the combination is really the solution of the given differential equation. □

This principle allows us to separate the differential equation into smaller problems.

Example 2.6.15. Solve the differential equation
\[ y'' + 4y = x + 2e^{-2x}. \]

ANSWER. General Solution \( y_h \). We first solve the homogeneous equation
\[ y'' + 4y = 0. \]

It has the general solution \( y_h(x) = c_1 \cos(2x) + c_2 \sin(2x). \)

Particular Solution \( y_p \). For a particular solution, we separate the differential equation into two pieces
\[ y'' + 4y = x \quad \text{and} \quad y'' + 4y = e^{-2x} \]
and solve each of them separately.

Problem 1. \( y'' + 4y = x \). Method of undetermined coefficients gives a solution \( y_{p_1}(x) = \frac{x}{4}. \)

Problem 2. \( y'' + 4y = e^{-2x} \). Method of undetermined coefficients gives a solution \( y_{p_2}(x) = \frac{e^{-2x}}{4}. \)

By the principle of superposition, thus, the original equation has a particular solution
\[ y_p(x) = y_{p_1}(x) + y_{p_2}(x) = \frac{x}{4} + \frac{e^{-2x}}{4} = \frac{1}{4} \left( x + \frac{1}{e^{2x}} \right). \]

General Solution. From Steps above, the general solution of the given differential equation is
\[ y(x) = y_h(x) + y_p(x) = c_1 \cos(2x) + c_2 \sin(2x) + \frac{1}{4} \left( x + \frac{1}{e^{2x}} \right). \]
2.6.4 Higher–Order Differential Equations.
As the last topic in this section, we consider a differential equation having order more than 2. Let us consider, for example, a 6th–order differential equation
\[ y^{(6)} - 4y^{(4)} + 2y' + 15y = 0. \]
Since it has constant coefficients, if we look for a solution of form \( e^{\lambda x} \), then we have the characteristic equation
\[ \lambda^6 - 4\lambda^4 + 2\lambda + 15 = 0, \]
which has 6 complex roots, say, \( p_1 \pm iq_1, p_2 \pm iq_2, \) and \( p_3 \pm iq_3. \) Hence, the general solution is
\[ y(x) = e^{p_1} (a_1 \cos (q_1 x) + a_2 \sin (q_1 x)) + e^{p_2} (b_1 \cos (q_2 x) + b_2 \sin (q_2 x)) + e^{p_3} (c_1 \cos (q_3 x) + c_2 \sin (q_3 x)), \]
where \( a_i, b_i \) and \( c_i, i = 1, 2, \) are arbitrary constants.

Is there a better way to solve the given higher–order differential equation? We use the series of change of variable as follows:
\[ z_1 = y, \quad z_2 = y', \quad z_3 = y'', \quad z_4 = y^{(3)}, \quad z_5 = y^{(4)}, \quad z_6 = y^{(5)}. \]
Then, the given differential equation is expressed by a system
\[ z'_1 = z_2, \quad z'_2 = z_3, \quad z'_3 = z_4, \quad z'_4 = z_5, \quad z'_5 = z_6, \quad z'_6 = 4z_5 - 2z_2 - 15z_1. \]

Remark 2.6.16 (ASIDE). Using the Matrix notation (which will be discussed in LINEAR ALGEBRA or CALCULUS 4), the system is formed as follows:
\[
\begin{pmatrix}
  z'_1 \\
  z'_2 \\
  z'_3 \\
  z'_4 \\
  z'_5 \\
  z'_6
\end{pmatrix} = \begin{pmatrix}
  0 & 1 & 0 & 0 & 0 & 0 \\
  0 & 0 & 1 & 0 & 0 & 0 \\
  0 & 0 & 0 & 1 & 0 & 0 \\
  0 & 0 & 0 & 0 & 1 & 0 \\
  -15 & -2 & 0 & 0 & 0 & 4 \\
\end{pmatrix} \begin{pmatrix}
  z_1 \\
  z_2 \\
  z_3 \\
  z_4 \\
  z_5 \\
  z_6
\end{pmatrix} \quad \text{or} \quad \mathbf{z}' = A\mathbf{z}',
\]
where \( A \) is the matrix and \( \mathbf{z} = (z_1, \ldots, z_6) \) and \( \mathbf{z}' = (z'_1, \ldots, z'_6) \) are vectors and \( \mathbf{z}' \) is the transpose of \( \mathbf{z} \). So simply speaking, the problem looks like
\[ \frac{d\mathbf{z}}{dx} = A\mathbf{z}, \]
which is a separable equation with the general solution, \( \mathbf{z} = e^{Ax}. \) How can we define and compute the matrix power of an exponential function? – References for Graduate Study in Math: Richard Bellman’s *Introduction to Matrix Analysis*, Vladimir Arnold’s *Ordinary Differential Equations*. Far and Far Beyond the level of this course.

§2.7 Application of Second–Order Differential Equations to a Mechanical System.
Skip. Please read the textbook.
Chapter 3

Laplace Transform

So far through Calculus courses and previous Chapters, we have seen many techniques on change of variable. In this chapter, we introduce a very special and powerful tool for solving a differential equation, which transforms an initial value problem itself to an algebra problem. The process is as follows.

1. **Initial Value Problem**
2. **Step 1. Laplace Transform**
3. **Step 2. Algebraic Methods**
4. **Step 3. Inverse Laplace Transform**
5. **Solution of Initial Value Problem**
6. **Solution of Algebra Problem**

### §3.1 Definition and Basic Properties.

**Definition 3.1.1.** The Laplace Transform \( \mathcal{L}[f] \) of \( f \) is a function defined by

\[
\mathcal{L}[f](s) = \int_0^\infty e^{-st} f(t) \, dt,
\]

for all \( s \) such that this integral converges.

The Laplace transform converts a function \( f \) into a new function \( \mathcal{L}[f] \). Often we use \( t \) as the independent variable of \( f \) and \( s \) as the independent variable of the function \( \mathcal{L}[f] \). It is convenient to use a capital letter for the Laplace transform, e.g.,

\[
F = \mathcal{L}[f], \quad G = \mathcal{L}[g], \quad H = \mathcal{L}[h],
\]

and so on.

**Example 3.1.2.** For \( f(t) = e^{at} \), with \( a \) any real number, find its Laplace transform \( \mathcal{L}[f](s) \).

**Answer.** By definition,

\[
\mathcal{L}[f](s) = F(s) = \int_0^\infty e^{-st} e^{at} \, dt = \int_0^\infty e^{(a-s)t} \, dt = \lim_{b \to \infty} \int_0^b e^{(a-s)t} \, dt
\]

\[
= \lim_{b \to \infty} \frac{1}{a-s} e^{(a-s)t} \bigg|_0^b = \lim_{b \to \infty} \frac{1}{a-s} \left( e^{(a-s)b} - 1 \right) = \frac{1}{a-s} \left[ e^{(a-s)b} - 1 \right] = \frac{1}{s-a},
\]

provided by \( a - s < 0 \), \( s > a \). Thus the Laplace transform of \( f(t) = e^{at} \) is \( F(s) = \frac{1}{s-a} \) defined for \( s > a \). \( \square \)

**Example 3.1.3.** For \( g(t) = \sin t \), find its Laplace transform \( \mathcal{L}[g](s) \).

**Answer.** By definition,

\[
\mathcal{L}[g](s) = G(s) = \int_0^\infty e^{-st} \sin t \, dt = \lim_{b \to \infty} \int_0^b e^{-st} \sin t \, dt
\]

\[
= \lim_{b \to \infty} -\frac{1}{s^2 + 1} \left[ e^{-bs} \cos b + se^{-bs} \sin b - 1 \right] = \frac{1}{s^2 + 1}.
\]

Thus \( G(s) = \frac{1}{s^2 + 1} \) defined for all real \( s \). \( \square \)
Whenever we need a Laplace transform of a function, should we compute and find it? Clearly, it is not efficient. So we use the table of the Laplace transform. See the Appendix 3.1.1.

**Theorem 3.1.4 (Linearity of Laplace Transform).** Suppose \( \mathcal{L}[f](s) \) and \( \mathcal{L}[g](s) \) are defined for \( s > a \) and \( \alpha \) and \( \beta \) are real numbers. Then, for \( s > a \),

1. (Constant Multiple) \( \mathcal{L}[\alpha f](s) = \alpha \mathcal{L}[f](s) \)
2. (Sum) \( \mathcal{L}[f + g](s) = \mathcal{L}[f](s) + \mathcal{L}[g](s) \)

The two formulas can be compressed into the following one formula,

\[
\mathcal{L}[\alpha f + \beta g](s) = \alpha \mathcal{L}[f](s) + \beta \mathcal{L}[g](s),
\]

which is called the **linearity property** of Laplace transform.

**Proof.** Since \( \mathcal{L}[f](s) \) and \( \mathcal{L}[g](s) \) are well defined for \( s > a \), the integrals \( \int_0^\infty e^{-st} f(t) \, dt \) and \( \int_0^\infty e^{-st} g(t) \, dt \) converge for \( s > a \). It implies that for \( s > a \),

\[
\mathcal{L}[\alpha f + \beta g](s) = \int_0^\infty e^{-st} (\alpha f(t) + \beta g(t)) \, dt
\]

\[
= \alpha \int_0^\infty e^{-st} f(t) \, dt + \beta \int_0^\infty e^{-st} g(t) \, dt = \alpha \mathcal{L}[f](s) + \beta \mathcal{L}[g](s).
\]

We can generalize the formula to finitely many functions. Under the appropriate circumstances, we have

\[
\mathcal{L} \left[ \sum_{k=1}^n \alpha_k f_k \right](s) = \alpha_1 \mathcal{L}[f_1] + \alpha_2 \mathcal{L}[f_2] + \cdots + \alpha_n \mathcal{L}[f_n] = \sum_{k=1}^n \alpha_k \mathcal{L}[f_k](s).
\]

Now we focus on functions. Let us refresh our memory on **Calculus I & II**.

1. On differentiation, can any function be differentiable? No! Then, what kind of function can be differentiable? We studied on the **differentiable function**. To be a differentiable function, it should be **at least continuous**.
2. On integration, can any function be integrable? No! Then, what kind of function can be integrable? We studied on the **integrable function**. To be an integrable function, it should be **continuous** (roughly speaking, because we can even integrate a function having finite number of jump discontinuities).

As we can see, the continuity is very important in differentiation and integration. Moreover, it goes to our subject, Laplace transform. Can any function have the Laplace transform? No! Then, what kind of function can have the Laplace transform? What is the requirement on the function? To have the Laplace transform, the function should be **at least piecewise continuous**.

**Definition 3.1.5 (Piecewise Continuity).** A function \( f \) is said to be **piecewise continuous** on the interval \([a, b]\) if there are points \( a = t_0 < t_1 < t_2 < \cdots < t_{n-1} < t_n = b \) such that

1. \( f \) is continuous on each open interval \((t_{j-1}, t_j)\), \( j = 1, 2, \ldots, n \) and
2. all of the following one–sided limits are finite,

\[
\lim_{t \to t_j^-} f(t), \quad \lim_{t \to t_j^+} f(t), \quad \lim_{t \to t_j^-} f(t), \quad \lim_{t \to t_j^+} f(t), \quad j = 1, 2, \ldots, n - 1.
\]
In other words, \( f \) is piecewise continuous on \([a, b]\) if it is continuous there except for a finite number of jump discontinuities. If \( f \) is piecewise continuous on \([a, b]\) for every \( b > a \), then \( f \) is said to be **piecewise continuous** on \([a, \infty)\).

See the figures 3.1 and 3.2. The figure 3.2 is the graph of the piecewise continuous function \( f(t) \) defined by

\[
f(t) = \begin{cases} 
    t^2 & \text{for } 0 \leq t < 2 \\
    2 & \text{at } t = 2 \\
    1 & \text{for } 2 < t \leq 3 \\
    -1 & \text{for } 3 < t \leq 4.
\end{cases}
\]

Figure 3.1: Example of piecewise continuous function on \([a, b]\).

Figure 3.2: Example of piecewise continuous function \( f(t) \) on \([0, 4]\).

If \( f \) is piecewise continuous on \([0, k]\) for every positive \( k \), then does its Laplace transform exist? The answer is no. Under the given condition on \( f \), \( e^{-st} f(t) \) is piecewise continuous on \([0, k]\) and so \( \int_0^k e^{-st} f(t) \, dt \) does exist. However, \( \lim_{k \to \infty} \int_0^k e^{-st} f(t) \, dt \) may not converge and the Laplace transform of \( f \) may not exist. For instance, \( f(t) = e^{t^2} \) is continuous everywhere but \( \int_0^\infty e^{-st} e^{t^2} \, dt \) does not converge for every real value of \( s \). For this reason, we need one more condition on \( f \) to guarantee the existence of its Laplace transform. The following theorem gives the conditions for the existence of the Laplace transform.
Theorem 3.1.6 (Existence of \( \mathcal{L}[f] \)).

1. Suppose \( f \) is piecewise continuous on \([0,k]\) for every positive \( k \).
2. Suppose there are numbers \( M \) and \( b \) such that \( |f(t)| \leq Me^{bt} \) for \( t \geq 0 \).

Then \( \int_0^\infty e^{-st}f(t)\,dt \) converges for \( s > b \) and hence the Laplace transform \( \mathcal{L}[f](s) \) of \( f \) is defined for \( s > b \).

**Rough Proof.** From the first condition, we deduce that \( e^{-st}f(t) \) is piecewise continuous on \([0,k]\) and so \( \int_0^k e^{-st}f(t)\,dt \) does exist. Now we use the second condition, which implies, for \( s \geq b \),

\[
|f(t)| \leq Me^{bt}, \quad e^{-st}|f(t)| \leq Me^{(b-s)t},
\]

but,

\[
\lim_{k \to \infty} \int_0^k Me^{(b-s)t}\,dt = \begin{cases} 
\frac{M}{s-b} & \text{for } b-s < 0, \text{ i.e., } s > b \\
\text{diverges} & \text{for } b-s \geq 0, \text{ i.e., } s \leq b
\end{cases}
\]

So by comparison, for \( s > b \),

\[
\left| \int_0^\infty e^{-st}f(t)\,dt \right| \leq \int_0^\infty e^{-st}|f(t)|\,dt \leq \int_0^\infty Me^{(b-s)t}\,dt = \frac{M}{s-b},
\]

i.e., for \( s > b \), the Laplace transform \( \mathcal{L}[f](s) = \int_0^\infty e^{-st}f(t)\,dt \) exists. \( \square \)

**Example 3.1.7.** Elementary functions having Laplace transforms are

1. polynomials
2. trigonometric functions \( \sin(at) \) and \( \cos(at) \)
3. exponential functions \( e^{at} \)

where \( a \) is a constant. \( \square \)

Let us consider the converse of the Theorem 3.1.6. If a function has the Laplace transform, then should it always satisfy both conditions in the Theorem 3.1.6? The answer is no. Even though a function has a Laplace transform, it may not satisfy both conditions. For instance, the function \( f(t) = t^{-1/2} \), for \( t > 0 \), has the Laplace transform,

\[
\mathcal{L}[f](s) = \int_0^\infty e^{-st}t^{-1/2}\,dt \stackrel{\text{Subst. } y = t^{1/2}}{=} 2 \int_0^\infty e^{-x^2}\,dx \stackrel{\text{Subst. } x = \sqrt{s}}{=} \frac{2}{\sqrt{s}} \int_0^\infty e^{-z^2}\,dz = \sqrt{\frac{\pi}{s}}
\]

where the fact \( \int_0^\infty e^{-z^2}\,dz = \frac{\sqrt{\pi}}{2} \) is used. However, the function \( f(t) = t^{-1/2} \) is not piecewise continuous on \([0,k]\) for every positive \( k \), precisely, \( \lim_{t \to 0^+} t^{-1/2} = \infty \) and so it does not satisfy the first condition in the Theorem 3.1.6.

Next, we introduce the inverse Laplace transform.

**Definition 3.1.8 (Inverse Laplace Transform).** Given a function \( G \), a function \( g \) such that \( \mathcal{L}[g] = G \) is called an inverse Laplace transform of \( G \). In this event, we write \( g = \mathcal{L}^{-1}[G] \).

**Example 3.1.9.** Since \( f(t) = e^{at} \) has the Laplace transform \( \mathcal{L}[f](s) = \frac{1}{s-a} \), so the inverse Laplace transform of \( F(s) = \frac{1}{s-a} \) is \( f(t) = e^{at} \), i.e.,

\[
\mathcal{L}[e^{at}] (s) = \frac{1}{s-a}, \quad e^{at} = \mathcal{L}^{-1}\left[\frac{1}{s-a}\right](t).
\]

Since \( f(t) = \sin t \) has the Laplace transform \( \mathcal{L}[f](s) = \frac{1}{s^2+1} \), so the inverse Laplace transform of \( F(s) = \frac{1}{s^2+1} \) is \( f(t) = \sin t \), i.e.,

\[
\mathcal{L}[\sin t](s) = \frac{1}{s^2+1}, \quad \sin t = \mathcal{L}^{-1}\left[\frac{1}{s^2+1}\right](t).
\]

\( \square \)
To find the inverse Laplace transform, we use the same table as the one for the Laplace transform.

Let us consider two functions,

\[ f(t) = e^{-t} \quad \text{and} \quad g(t) = \begin{cases} e^{-t} & \text{for } t \neq 3 \\ 0 & \text{for } t = 3. \end{cases} \]

We observe \( \mathcal{L}[f](s) = \frac{1}{s+1} = \mathcal{L}[g](s) \) for \( s > -1 \), that is, \( f \) and \( g \) have the same Laplace transform. Then which one of \( f \) and \( g \) will be the inverse Laplace transform of \( \frac{1}{s+1} \)? That is,

\[ \mathcal{L}^{-1}\left[\frac{1}{s+1}\right](t) = f(t)? \quad \text{or} \quad \mathcal{L}^{-1}\left[\frac{1}{s+1}\right](t) = g(t)? \]

The following theorem gives the answer to this question.

**Theorem 3.1.10 (LERCH’S THEOREM).** Let \( f \) and \( g \) be continuous on \([0,\infty)\) and suppose \( \mathcal{L}[f] = \mathcal{L}[g] \). Then \( f = g \).

That is, if two continuous functions have the same Laplace transform, then they should be same. Thus, we can deduce the uniqueness of the inverse Laplace transform, provided the inverse Laplace transform is continuous. Notice that the function \( g(t) \) presented before the theorem is not continuous. Because of the linearity of the Laplace transform, its inverse is also linear.

**Theorem 3.1.11 (LINEARITY OF INVERSE LAPLACE TRANSFORM).** If \( \mathcal{L}^{-1}[F] = f \) and \( \mathcal{L}^{-1}[G] = g \) and \( \alpha \) and \( \beta \) are real numbers, then

\[ \mathcal{L}^{-1}[\alpha F + \beta G] = \alpha \mathcal{L}^{-1}[F] + \beta \mathcal{L}^{-1}[G] = \alpha f + \beta g. \]
### 3.1.1 Appendix (Table of Laplace Transform).

<table>
<thead>
<tr>
<th>$f(t)$</th>
<th>$F(s) = \mathcal{L}<a href="s">f</a>$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$\frac{1}{s}$</td>
</tr>
<tr>
<td>$t$</td>
<td>$\frac{1}{s^2}$</td>
</tr>
<tr>
<td>$t^n$</td>
<td>$\frac{n!}{s^{n+1}}$</td>
</tr>
<tr>
<td>$e^{at}$</td>
<td>$\frac{1}{s - a}$</td>
</tr>
<tr>
<td>$te^{at}$</td>
<td>$\frac{1}{(s - a)^2}$</td>
</tr>
<tr>
<td>$\sin(at)$</td>
<td>$\frac{a}{s^2 + a^2}$</td>
</tr>
<tr>
<td>$\cos(at)$</td>
<td>$\frac{s}{s^2 + a^2}$</td>
</tr>
<tr>
<td>$t \sin(at)$</td>
<td>$\frac{2as}{(s^2 + a^2)^2}$</td>
</tr>
<tr>
<td>$t \cos(at)$</td>
<td>$\frac{s^2 - a^2}{(s^2 + a^2)^2}$</td>
</tr>
<tr>
<td>$e^{at}\sin(bt)$</td>
<td>$\frac{b}{(s - a)^2 + b^2}$</td>
</tr>
<tr>
<td>$e^{at}\cos(bt)$</td>
<td>$\frac{s - a}{(s - a)^2 + b^2}$</td>
</tr>
</tbody>
</table>

Here $n! = n(n - 1)(n - 2) \cdots 3(2)(1)$ and $0! = 1$. 


Section 3.2 Solution of Initial Value Problem Using the Laplace Transform.

We use the Laplace transform to solve an initial value problem. Since a derivative \( y'(t) \) of a function \( y(t) \) is also a function, with appropriate conditions, we can find the Laplace transform \( \mathcal{L}[y'](s) \) of the derivative \( y'(t) \) and thus we can change the differential equation of \( y(t) \) into the equation of its Laplace transform \( \mathcal{L}[y](s) \). For this purpose, we need the following theorem.

**Theorem 3.2.1 (LAPLACE TRANSFORM OF DERIVATIVE).** Suppose \( f \) is continuous on \([0, \infty)\) and has the limit \( \lim_{k \to \infty} e^{-sk}f(k) = 0 \) for \( s > 0 \) and \( f' \) is piecewise continuous on \([0, k)\) for every positive \( k \). Then, with \( F(s) = \mathcal{L}[f](s) \),

\[
\mathcal{L}[f'](s) = sF(s) - f(0).
\]

**Proof.** The Integration by Parts formula with \( u = e^{-st} \) and \( dv = f'(t)dt \) implies, for \( k > 0 \),

\[
\int_0^k e^{-st}f'(t)\,dt = \left[ uv \right]_0^k - \int_0^k v\,du = \left[ e^{-st}f(t) \right]_0^k + s \int_0^k e^{-st}f(t)\,dt = e^{-ks}f(k) - f(0) + s \int_0^k e^{-st}f(t)\,dt.
\]

In the case that \( f \) has a jump discontinuity at \( t = 0 \), the formula becomes \( \mathcal{L}[f'](s) = sF(s) - f(0^+) \), where \( f(0^+) = \lim_{t \to 0^+} f(t) \).

Now we apply the Laplace transform to the higher derivatives. Recall that \( f^{(j)} \) means the \( j \)th derivative of \( f \) and \( f^{(0)} = f \).

**Theorem 3.2.2 (LAPLACE TRANSFORM OF HIGHER DERIVATIVE).** Suppose \( f, f', \ldots, f^{(n-1)} \) are continuous on \([0, \infty)\), \( \lim_{k \to \infty} e^{-sk}f^{(j)}(k) = 0 \) for \( s > 0 \) and \( j = 1, 2, \ldots, n-1 \) and \( f^{(n)} \) is piecewise continuous on \([0, k)\) for every positive \( k \). Then, with \( F(s) = \mathcal{L}[f](s) \),

\[
\mathcal{L}[f^{(n)}](s) = s^nF(s) - \sum_{i=1}^{n-1} s^{n-i}f^{(i-1)}(0),
\]

The Theorems 3.2.1 and 3.2.2 imply the following corollary.

**Corollary 3.2.3.**

\[
\mathcal{L}[f'](s) = sF(s) - f(0),
\]

\[
\mathcal{L}[f''](s) = s^2F(s) - sf(0) - f'(0),
\]

\[
\mathcal{L}[f'''](s) = s^3F(s) - s^2f(0) - sf'(0) - f''(0).
\]

Now we are ready to solve an initial value problem using the Laplace transform.

**Example 3.2.4.** Solve the initial value problem \( y'(t) = 3 \) with \( y(0) = 5 \).

**Answer.** Applying the Laplace transform to the differential equation \( y'(t) = 3 \), we get

\[
\mathcal{L}[y'](s) = \mathcal{L}[3](s) = 3\mathcal{L}[1](s), \quad sY(s) - y(0) = \frac{3}{s}, \quad sY(s) - 5 = \frac{3}{s}, \quad Y(s) = \frac{3}{s^2} + \frac{5}{s}.
\]

Applying the inverse Laplace transform to the resulting function \( Y(s) = \frac{3}{s^2} + \frac{5}{s} \) for \( s > 0 \), we get

\[
y(t) = \mathcal{L}^{-1}[Y](t) = \mathcal{L}^{-1}\left[ \frac{3}{s^2} + \frac{5}{s} \right](t) = 3\mathcal{L}^{-1}\left[ \frac{1}{s^2} \right](t) + 5\mathcal{L}^{-1}\left[ \frac{1}{s} \right](t) = 3t + 5.
\]
Example 3.2.5. Solve the initial value problem \( y' - 4y = 0, \ y(0) = 1. \)

**ANSWER.** Applying the Laplace transform to the differential equation, we get

\[
\mathcal{L} [y' - 4y] (s) = \mathcal{L} [0](s), \quad \mathcal{L} [y'] (s) - 4\mathcal{L} [y] (s) = \mathcal{L} [0](s), \\
sY(s) - y(0) - 4Y(s) = 0, \quad (s-4)Y(s) = 1, \quad Y(s) = \frac{1}{s-4} \quad (s > 4).
\]

Applying the inverse Laplace transform to the resulting function \( Y(s) = \frac{1}{s-4} \) for \( s > 4, \)

\[
y(t) = \mathcal{L}^{-1} [Y](t) = \mathcal{L}^{-1} \left[ \frac{1}{s-4} \right] (t) = e^{4t}. \quad \square
\]

Example 3.2.6. Solve the initial value problem \( y'' + 4y' + 3y = e^t, \ y(0) = 0, \ y'(0) = 2. \)

**ANSWER.** Applying the Laplace transform to the differential equation, we get

\[
\mathcal{L} [e^t] (s) = \mathcal{L} [y'' + 4y' + 3y] (s) = \mathcal{L} [y''] (s) + 4\mathcal{L} [y'] (s) + 3\mathcal{L} [y] (s)
\]

\[
\frac{1}{s-1} = s^2Y(s) - sy(0) - y'(0) + 4[sY(s) - y(0)] + 3Y(s)
\]

\[
= s^2Y(s) - s(0) - 2 + 4[sY(s) - y(0)] + 3Y(s)
\]

\[
\frac{2s-1}{s-1} = \frac{1}{s-1} + 2 = (s^2 + 4s + 3)Y
\]

\[
Y(s) = \frac{2s-1}{(s-1)(s^2 + 4s + 3)} \quad (s > 1).
\]

Before we apply the inverse Laplace transform, we need to separate the fraction partially, i.e., **Partial Fraction Technique** in **CALCULUS I** should be used, because the table on the Laplace transform does not provide the (inverse) Laplace transform for the fraction \( \frac{2s-1}{(s-1)(s^2 + 4s + 3)} \), which one can check by yourself.

We separate the fraction as follows:

\[
\frac{2s-1}{(s-1)(s^2 + 4s + 3)} = \frac{2s-1}{(s-1)(s+1)(s+3)} = \frac{A}{s-1} + \frac{B}{s+1} + \frac{C}{s+3},
\]

where \( A, B \) and \( C \) are constants to be determined. Multiplying the equation by \( (s-1)(s+1)(s+3) \),

\[
2s-1 = A(s+1)(s+3) + B(s-1)(s+3) + C(s-1)(s+1).
\]

Putting \( s = 1, \ s = -1 \) and \( s = -3 \), we have respectively

\[
2(1) - 1 = A(2)(4) + B(0) + C(0), \quad 1 = 8A, \quad A = \frac{1}{8}
\]

\[
2(-1) - 1 = A(0) + B(-2)(2) + C(0), \quad -3 = -4B, \quad B = \frac{3}{4}
\]

\[
2(-3) - 1 = A(0) + B(0) + C(-4)(-2), \quad -7 = 8C, \quad C = -\frac{7}{8}.
\]

\[
Y(s) = \frac{2s-1}{(s-1)(s^2 + 4s + 3)} = \frac{A}{s-1} + \frac{B}{s+1} + \frac{C}{s+3} = \frac{1}{8s-1} + \frac{3}{4s+1} - \frac{7}{8s+3}.
\]

Now applying the inverse Laplace transform to the resulting function for \( s > 1, \)

\[
y(t) = \mathcal{L}^{-1} [Y](t) = \mathcal{L}^{-1} \left[ \frac{1}{8s-1} + \frac{3}{4s+1} - \frac{7}{8s+3} \right] (t)
\]

\[
= \frac{1}{8} \mathcal{L}^{-1} \left[ \frac{1}{s-1} \right] (t) + \frac{3}{4} \mathcal{L}^{-1} \left[ \frac{1}{s+1} \right] (t) - \frac{7}{8} \mathcal{L}^{-1} \left[ \frac{1}{s+3} \right] (t) = \frac{1}{8} e^t + \frac{3}{4} e^{-t} - \frac{7}{8} e^{-3t}. \quad \square
\]
Example 3.2.7. Solve the initial value problem $y'' + y = t$, $y(0) = 1$, $y'(0) = 0$.

**Answer.** Applying the Laplace transform to the differential equation, we get

$$
L[y'' + y](s) = L[t](s), \quad L[y'' + y](s) + L[y](s) = L[t](s),
$$

$$
s^2Y(s) - sy(0) - y'(0) + Y(s) = \frac{1}{s^2}, \quad s^2Y - s - 0 + Y = \frac{1}{s^2},
$$

$$
(s^2 + 1)Y = \frac{1}{s^2} + s, \quad Y(s) = \frac{1}{s^2(s^2 + 1)} + \frac{s}{s^2 + 1} \quad (s > 0).
$$

Applying the inverse Laplace transform to the resulting function for $s > 0$, we get

$$
y(t) = L^{-1}[Y](t) = L^{-1}\left[\frac{1}{s^2(s^2 + 1)} + \frac{s}{s^2 + 1}\right](t)
$$

$$
= L^{-1}\left[\frac{1}{s^2(s^2 + 1)}\right](t) + L^{-1}\left[\frac{s}{s^2 + 1}\right](t) = t - \sin t + \cos t. \quad \square
$$

An interesting consequence of the Laplace transform can be found as follows. We recall

$$
L[f'] = sL[f] - f(0).
$$

Suppose $f(t)$ is defined by

$$
f(t) = \int_0^t g(\tau) \, d\tau.
$$

Then $f'(t) = g(t)$ and $f(0) = 0$. So we observe

$$
L[g] = L[f'] = sL[f] - f(0) = sL[f] - 0 = sL[f], \quad L[f] = \frac{1}{s}L[g], \quad L\left[\int_0^t g(\tau) \, d\tau\right] = \frac{1}{s}L[g].
$$

It enables us to take the Laplace transform of a function defined by an integral.
§3.3 Shifting Theorems and the Heaviside Function.
In this section, we discuss four topics: (i) the first shifting theorem, (ii) the heaviside function and pulses, (iii) the first shifting theorem, and (iv) analysis of electrical circuits.

Let us compare two graphs of $y = 2t$ and $y = 2(t - 5)$. We observe the graph of $y = 2(t - 5)$ is obtained by shifting the graph of $y = 2t$ by 5 units along the positive $t$--axis, i.e., by moving parallelly the graph of $y = 2t$ by 5 units along the positive $t$--axis. See the figure 3.3.

![Figure 3.3: Graphs of $y = 2t$ and $y = 2(t - 5)$.](image)

In general, when we shift the graph $y = f(t)$ by $a$ units along the positive $t$--axis, we obtain the graph of $y = f(t - a)$. Thanks to the shifting relation, once we know the properties of $y = f(t)$, we can deduce the properties of $y = f(t - a)$ easily. We will see how greatly this shifting technique works in the Laplace transform.

3.3.1 The First Shifting Theorem.

**Theorem 3.3.1 (First Shifting Theorem).** Let $\mathcal{L}[f](s) = F(s)$ for $s > b \geq 0$. Let $a$ be any number. Then

$$\mathcal{L}[e^{at}f(t)](s) = F(s - a).$$

for $s > a + b$. Moreover, the inverse Laplace transform yields $\mathcal{L}^{-1}[F(s - a)] = e^{at}f(t)$, i.e.,

$$\mathcal{L}^{-1}[F(s - a)] = e^{at}\mathcal{L}^{-1}[F(s)].$$

The theorem says that the Laplace transform of $e^{at}f(t)$ is exactly same as the Laplace transform of $f(t)$ shifted $a$ units to the positive $s$--axis.

**Proof.** A simple computation shows

$$\mathcal{L}[e^{at}f(t)](s) = \int_0^\infty e^{at}e^{-st}f(t)\,dt = \int_0^\infty e^{-(s-a)t}f(t)\,dt = F(s - a) \quad (s - a > b, \ s > a + b).$$

**Example 3.3.2.** Since $\mathcal{L}[\cos(bt)] = \frac{s}{s^2 + b^2}$, so $\mathcal{L}[e^{at}\cos(bt)] = \frac{s-a}{(s-a)^2 + b^2}$.

**Example 3.3.3.** Since $\mathcal{L}[t^3] = \frac{6}{s^4}$, so $\mathcal{L}[t^3e^{7t}] = \frac{6}{(s-7)^4}$.

**Exercise 3.3.4.** Find the Laplace transform of $f(t) = e^{-t}\cos(2t) - 2\sin(3t)$. (Final Exam of Fall 2009)
Remark 3.3.5 (Vertex Form of Quadratic Function from Calculus I). The complete square technique for a quadratic function \( f(s) = as^2 + bs + c \) with constants \( a, b \) and \( c \), is very useful to find the inverse Laplace transform of the shifted function. The question is how to convert \( f(s) \) in the form of \( f(s) = a(s - \Box)^2 + \Delta \)?

From Calculus I, \( f(s) = as^2 + bs + c \) has the critical number \( s = -\frac{b}{2a} \). (To refresh your memory, \( f'(s) = 2as + b = 0 \) at \( s = -\frac{b}{2a} \), which is the critical number.) At the critical number, \( f \) has the value

\[
f\left(-\frac{b}{2a}\right) = a\left(-\frac{b}{2a}\right)^2 + b\left(-\frac{b}{2a}\right) + c = -\frac{b^2 - 4ac}{4a}.
\]

It gives the following equation

\[
f(s) = as^2 + bs + c = a\left(s + \frac{b}{2a}\right)^2 - \frac{b^2 - 4ac}{4a},
\]

which is called the vertex\(^1\) form of the quadratic function \( f \) with the vertex point \((-\frac{b}{2a}, -\frac{b^2 - 4ac}{4a})\).

Example 3.3.6. Find \( \mathcal{L}^{-1}\left[\frac{4}{s^2 + 4s + 20}\right] \).

**Answer.** First we find the vertex form of the denominator \( s^2 + 4s + 20 \). Using the technique above, we deduce

\[
s^2 + 4s + 20 = (s + 2)^2 + 4^2, \quad \frac{4}{s^2 + 4s + 20} = \frac{4}{(s + 2)^2 + 4^2}.
\]

It enables us to think of a function

\[
F(s + 2) = \frac{4}{(s + 2)^2 + 4^2}, \quad F(s) = \frac{4}{s^2 + 4^2}.
\]

Now we apply the shifting theorem,

\[
\mathcal{L}^{-1}[F(s + 2)] = e^{-2t}\mathcal{L}^{-1}[F(s)] = e^{-2t}\sin(4t), \quad \text{i.e.,} \quad \mathcal{L}^{-1}\left[\frac{4}{s^2 + 4s + 20}\right] = e^{-2t}\sin(4t). \quad \Box
\]

Example 3.3.7. Compute \( \mathcal{L}^{-1}\left[\frac{3s - 1}{s^2 - 6s + 2}\right] \).

**Answer.** First we find the vertex form of the denominator \( s^2 - 6s + 2 \). By the technique above, we deduce

\[
s^2 - 6s + 2 = (s - 3)^2 - 7, \quad \frac{3s - 1}{s^2 - 6s + 2} = \frac{3s - 1}{(s - 3)^2 - 7} = \frac{3(s - 3)}{(s - 3)^2 - 7} + \frac{8}{(s - 3)^2 - 7} = G(s - 3) + K(s - 3),
\]

where \( G(s) = \frac{3s}{s^2 - 7} \), \( K(s) = \frac{8}{s^2 - 7} \).

Now we apply the shifting theorem,

\[
\mathcal{L}^{-1}\left[\frac{3s - 1}{s^2 - 6s + 2}\right] = \mathcal{L}^{-1}[G(s - 3) + K(s - 3)] = \mathcal{L}^{-1}[G(s - 3)] + \mathcal{L}^{-1}[K(s - 3)]
\]

\[
e^{3t}\mathcal{L}^{-1}[G(s)] + e^{3t}\mathcal{L}^{-1}[K(s)] = e^{3t}\left(\mathcal{L}^{-1}[G(s)] + \mathcal{L}^{-1}[K(s)]\right)
\]

\[
e^{3t}\left(\mathcal{L}^{-1}\left[\frac{3s}{s^2 - 7}\right] + 8\mathcal{L}^{-1}\left[\frac{1}{s^2 - 7}\right]\right) = e^{3t}\left(3\cosh(\sqrt{7}t) + \frac{8}{\sqrt{7}}\sinh(\sqrt{7}t)\right). \quad \Box
\]

\(^1\)(in mathematics) the point where two lines meet to form an angle, or the point that is opposite the base of a shape.
3.3.2 The Heaviside Function and Pulses.
We introduce a special function to deal with a nonhomogeneous differential equation whose forcing function has finite number of jump discontinuities. We recall that \( f \) has a jump discontinuity at \( a \) if both \( \lim_{t \to a^-} f(t) \) and \( \lim_{t \to a^+} f(t) \) exist and are finite but unequal. The magnitude of the jump discontinuity is defined by the width of the gap, i.e., the width is
\[
\left| \lim_{t \to a^-} f(t) - \lim_{t \to a^+} f(t) \right|.
\]
See the figure 3.4.

**Definition 3.3.8.** The **Heaviside function** \( H \) is defined by
\[
H(t) = \begin{cases} 
0 & \text{for } t < 0 \\
1 & \text{for } t \geq 0.
\end{cases}
\]
See the figure 3.5.

For any number \( a \), the Heaviside function shifted \( a \) units to the right is easily obtained by
\[
H(t - a) = \begin{cases} 
0 & \text{for } t < a \\
1 & \text{for } t \geq a.
\end{cases}
\]
See the figure 3.6.

By the definition of the shifted Heaviside function, we can obtain
\[
H(t - a)g(t) = \begin{cases} 
0 & \text{for } t < a \\
g(t) & \text{for } t \geq a.
\end{cases}
\]
Example 3.3.9. For \( g(t) = \cos t \) for all \( t \) and with \( a = \pi \), we have

\[
H(t - \pi)g(t) = H(t - \pi)\cos t = \begin{cases} 
0 & \text{for } t < \pi \\
\cos t & \text{for } t \geq \pi.
\end{cases}
\]

See the figure 3.7.

Definition 3.3.10. A \textbf{pulse} is a function of the form

\[
H(t - a) - H(t - b) = \begin{cases} 
0 & \text{for } t < a \\
1 & \text{for } a \leq t < b \\
0 & \text{for } t \geq b
\end{cases} \quad (a < b).
\]

See the figure 3.8

Example 3.3.11. For \( g(t) = e^t \) for all \( t \) with \( a = 1 \) and \( b = 2 \), we have

\[
[H(t - a) - H(t - b)]g(t) = [H(t - 1) - H(t - 2)]e^t = \begin{cases} 
0 & \text{for } t < 1 \\
e^t & \text{for } 1 \leq t < 2 \\
0 & \text{for } t \geq 2.
\end{cases}
\]

See the figure 3.9.
Next we consider the function of the form \( H(t-a)g(t-a) \), explicitly,

\[
H(t-a)g(t-a) = \begin{cases} 
0 & \text{for } t < a \\
g(t-a) & \text{for } t \geq a.
\end{cases}
\]

So we observe the graph of \( H(t-a)g(t-a) \) is obtained by the graph of \( g(t) \) shifted \( a \) units for \( t \geq a \) and zero for \( t < a \).

**Example 3.3.12.** For \( g(t) = t^2 \) and \( a = 2 \),

\[
H(t-a)g(t-a) = H(t-2)(t-2)^2 = \begin{cases} 
0 & \text{for } t < 2 \\
(t-2)^2 & \text{for } t \geq 2.
\end{cases}
\]

See the figure 3.10.

It is important to understand the difference between \( g(t) \), \( H(t-a)g(t) \) and \( H(t-a)g(t-a) \). See the figure 3.11 which is about the graphs of \( y = t^2 \), \( y = t^2H(t-3) \) and \( y = (t-3)^2H(t-3) \).

\( \square \)

**3.3.3 The Second Shifting Theorem.**

Sometimes \( H(t-a)g(t-a) \) is referred to as a **shifting function** and the second shifting theorem deals with the Laplace transform of the shifting function.

**Theorem 3.3.13 (Second Shifting Theorem).** Let \( \mathcal{L}[f](s) = F(s) \). Then, \( \mathcal{L}[H(t-a)f(t-a)](s) = e^{-as}F(s) \) for \( s > b \), i.e.,

\[
\mathcal{L}[H(t-a)f(t-a)](s) = e^{-as}\mathcal{L}[f](s).
\]
Moreover, the inverse Laplace transform yields \( \mathcal{L}^{-1}[e^{-as}F(s)](t) = H(t-a)f(t-a) \), i.e.,
\[
\mathcal{L}^{-1}[e^{-as}F(s)](t) = H(t-a)\mathcal{L}^{-1}[F](t-a).
\]

**Proof.** A simple computation shows
\[
\mathcal{L}[H(t-a)f(t-a)](s) = \int_0^{\infty} e^{-st} H(t-a)f(t-a) \, dt = \int_a^{\infty} e^{-st} f(t-a) \, dt \\
= e^{-as} \int_0^{\infty} e^{-sw} f(w) \, dw = e^{-as} \mathcal{L}[f](s).
\]

**Example 3.3.14.** Find the Laplace transform of \( H(t-a) \).

**Answer.** Let \( f(t) = 1 \) for all \( t \). Then, \( f(t-a) = 1 \) for any \( a \) and \( t \) and so we have \( H(t-a) = H(t-a)1 = H(t-a)f(t-a) \). By the Theorem 3.3.13, we get
\[
\mathcal{L}[H(t-a)](s) = e^{-as} \mathcal{L}[f](s) = e^{-as} \mathcal{L}[1](s) = \frac{1}{s} e^{-as}.
\]

**Example 3.3.15.** Compute \( \mathcal{L}[g] \) where
\[
g(t) = \begin{cases} 
0 & \text{for } 0 \leq t < 2 \\
36 & \text{for } t \geq 2.
\end{cases}
\]

**Answer.** We observe that \( g(t) = H(t-2)(t^2+1) \). Since the right-hand side has the Heaviside function shifted 2 units, we change the multiple \( t^2+1 \) as follows:
\[
t^2+1 = (t-2+2)^2+1 = (t-2)^2+4(t-2)+5.
\]
Then we get
\[
g(t) = H(t-2)(t^2+1) = H(t-2)\left[(t-2)^2+4(t-2)+5\right] \\
= H(t-2)(t-2)^2+4H(t-2)(t-2)+5H(t-2).
\]
By the Second Shifting Theorem 3.3.13, we deduce
\[
\mathcal{L}[g] = \mathcal{L}\left[H(t-2)\left[(t-2)^2+4(t-2)+5\right]\right]
\]
To get the solution of the initial value problem, we need to apply the inverse Laplace transform, \( \mathcal{L}^{-1} \), to determine the solution.

**Example 3.3.19.** Now we are ready to solve certain initial value problem involving discontinuous forcing functions.

**Exercise 3.3.18.** Find the inverse Laplace transform of the function

\[
g(t) = \begin{cases} 
  t & \text{for } 0 \leq t < 2 \\
  2e^{3t} & \text{for } t \geq 2.
\end{cases}
\]

**Example 3.3.17.** Compute \( \mathcal{L}^{-1} \left[ \frac{se^{-3s}}{s^2+4} \right] \).

**Answer.** By the Second Shifting Theorem 3.3.13, we have

\[
\mathcal{L}^{-1} \left[ \frac{se^{-3s}}{s^2+4} \right] = H(t-3) \mathcal{L}^{-1} \left[ \frac{s}{s^2+4} \right] (t-3) = H(t-3) \cos(2(t-3)),
\]

because \( \mathcal{L}^{-1} \left[ \frac{s}{s^2+4} \right] = \cos(2t) \) by table of Laplace transforms.

**Exercise 3.3.16.** Find the Laplace transform of the function

\[
F(s) = \frac{e^{-2s}}{s^2+5s+2}.
\]

Now we are ready to solve certain initial value problem involving discontinuous forcing functions.

**Example 3.3.19.** Solve the initial value problem \( y'' + 4y = f(t), y(0) = 0 = y'(0) \), where

\[
f(t) = \begin{cases} 
  0 & \text{for } t < 3 \\
  t & \text{for } t \geq 3.
\end{cases}
\]

**Answer.** The problem can be rewritten by \( y'' + 4y = H(t-3)t, y(0) = 0 = y'(0) \). We apply the Laplace transform,

\[
\mathcal{L}[y'' + 4y] = \mathcal{L}[H(t-3)t], \quad \mathcal{L}[y'] + 4\mathcal{L}[y] = \mathcal{L}[H(t-3)t],
\]

\[
s^2Y - sy(0) - y'(0) + 4Y = \mathcal{L}[H(t-3)t], \quad (s^2 + 4)Y = \mathcal{L}[H(t-3)t], \quad Y = \frac{1}{s^2+4} \mathcal{L}[H(t-3)t].
\]

To determine \( \mathcal{L}[H(t-3)t] \), we use the Second Shifting Theorem 3.3.13,

\[
H(t-3)t = H(t-3)[t-3+3] = H(t-3)(t-3) + 3H(t-3)
\]

\[
\mathcal{L}[H(t-3)t] = \mathcal{L}[H(t-3)(t-3) + 3H(t-3)] = \mathcal{L}[H(t-3)(t-3)] + 3\mathcal{L}[H(t-3)]
\]

\[
e^{-3s} \mathcal{L}[t] + 3e^{-3t} \mathcal{L}[1] = e^{-3s} \left( \mathcal{L}[t] + 3\mathcal{L}[1] \right) = e^{-3s} \left( \frac{1}{s^2} + \frac{3}{s} \right).
\]

Hence, we have

\[
Y(s) = \frac{1}{s^2+4} \mathcal{L}[H(t-3)t] = \frac{1}{s^2+4} e^{-3s} \left[ \frac{1}{s^2} + \frac{3}{s} \right] = \frac{3s+1}{s^2(s^2+4)} e^{-3s}.
\]

To get the solution of the initial value problem, we need to apply the inverse Laplace transform,

\[
y(t) = \mathcal{L}^{-1}[Y(s)](t) = \mathcal{L}^{-1} \left[ \frac{3s+1}{s^2(s^2+4)} e^{-3s} \right](t).
\]
In order to use the table of the Laplace transform, we separate the function by using the partial fraction technique,

\[
\frac{3s + 1}{s^2(s^2 + 4)} e^{-3s} = \frac{3}{4} \frac{1}{s} e^{-3s} - \frac{3}{4} \frac{s}{s^2 + 4} e^{-3s} + \frac{1}{4} \frac{1}{s^2} e^{-3s} - \frac{1}{4} \frac{1}{s^2 + 4} e^{-3s}.
\]

Applying the inverse Laplace transform, we deduce

\[
y(t) = \mathcal{L}^{-1}[Y(s)](t) = \mathcal{L}^{-1}\left[\frac{3s + 1}{s^2(s^2 + 4)} e^{-3s}\right](t)
\]

\[
= \frac{3}{4} \mathcal{L}^{-1}\left[\frac{1}{s} e^{-3s}\right] - \frac{3}{4} \mathcal{L}^{-1}\left[\frac{s}{s^2 + 4} e^{-3s}\right] + \frac{1}{4} \mathcal{L}^{-1}\left[\frac{1}{s^2} e^{-3s}\right] - \frac{1}{4} \mathcal{L}^{-1}\left[\frac{1}{s^2 + 4} e^{-3s}\right]
\]

\[
= \frac{3}{4} H(t - 3) - \frac{3}{4} H(t - 3) \cos(2(t - 3)) + \frac{1}{4} H(t - 3)(t - 3) - \frac{1}{4} H(t - 3) \frac{1}{2} \sin(2(t - 3))
\]

\[
= \frac{1}{4} H(t - 3)(t - 3) - \frac{1}{8} H(t - 3) [6 \cos(2(t - 3)) + \sin(2(t - 3))]
\]

\[
= \begin{cases} 
0 & \text{for } t < 3 \\
\frac{t}{4} - \frac{1}{8} [6 \cos(2(t - 3)) + \sin(2(t - 3))] & \text{for } t \geq 3.
\end{cases}
\]

See the figure 3.12.

![Figure 3.12: Graph of the solution](image)

We observe that the solution is differentiable everywhere even though the function \(f\) in the differential equation had a jump discontinuity at 3. This behavior is typical of the initial value problem having a discontinuous forcing function. In general, if a differential equation has order \(n\) and \(\Phi\) is a solution. Then \(\Phi\) and its first \(n - 1\) derivatives will be continuous, while the \(n^{th}\) derivative will have the jump discontinuity wherever \(f\) does and these jump discontinuities will agree in magnitude with the corresponding jump discontinuities of \(f\).

**Exercise 3.3.20.** Use the Laplace transform to solve \(y'' - 4y = 5, y(0) = 1\) and \(y'(0) = 0\).

As the last example under the topic, let us consider a function having two jump discontinuities.

**Example 3.3.21.** Express the given function \(f\) in terms of the pulse:

\[
f(t) = \begin{cases} 
0 & \text{for } t < 2 \\
t - 1 & \text{for } 2 \leq t < 3 \\
-4 & \text{for } t \geq 3.
\end{cases}
\]

**Answer.** First, we consider the function \(f\) only on \([2, 3]\). On that interval, \(f\) is expressed by the pulse \(H(t - 2) - H(t - 3)\) multiplied by \(t - 1\), i.e., \(f(t) = [H(t - 2) - H(t - 3)](t - 1)\) on \([2, 3]\). On the interval \([3, \infty)\), the function \(f\) is expressed by \(f(t) = -4H(t - 3)\). Therefore, we conclude

\[
f(t) = [H(t - 2) - H(t - 3)](t - 1) - 4H(t - 3).
\]
3.3.4 Analysis of Electrical Circuits.
Skip. Please read the textbook.
§3.4 Convolution.
So far in the knowledge on mathematics, we have studied operations on functions such as addition \((f + g)\), subtraction \((f - g)\), multiplication \((cf \text{ and } fg)\), division \((f/g)\) and composition \(f \circ g\). In this section, we introduce another special operation on functions called the convolution \((f \ast g)\). Why do we study the convolution in this chapter “Laplace Transform”? One of the reasons is the Convolution Theorem.

**Definition 3.4.1.** If \(f\) and \(g\) are defined on \([0, \infty)\), then the convolution \(f \ast g\) of \(f\) with \(g\) is the function defined by

\[
(f \ast g)(t) = \int_{0}^{t} f(t - \tau)g(\tau) \, d\tau \quad (t \geq 0).
\]

Is the convolution commutative? That is, \(f \ast g = g \ast f\)? The answer is yes, which will be discussed soon. More important property is the following theorem.

**Theorem 3.4.2 (Convolution Theorem).** If \(f \ast g\) is defined, then

\[
\mathcal{L}[f \ast g] = \mathcal{L}[f] \cdot \mathcal{L}[g].
\]

**Proof.** Let \(F = \mathcal{L}[f]\) and \(G = \mathcal{L}[g]\). Then

\[
F(s)G(s) = F(s) \int_{0}^{\infty} e^{-st}g(\tau) \, d\tau = \int_{0}^{\infty} F(s)e^{-st}g(\tau) \, d\tau.
\]

We recall

\[
e^{-st}F(s) = \mathcal{L}[H(t - \tau)f(t - \tau)](s).
\]

It implies

\[
F(s)G(s) = \int_{0}^{\infty} F(s)e^{-st}g(\tau) \, d\tau = \int_{0}^{\infty} \mathcal{L}[H(t - \tau)f(t - \tau)](s)g(\tau) \, d\tau.
\]

The definition of the Laplace transform gives

\[
\mathcal{L}[H(t - \tau)f(t - \tau)](s) = \int_{0}^{\infty} e^{-st}H(t - \tau)f(t - \tau) \, dt.
\]

So the function \(F(s)G(s)\) becomes

\[
F(s)G(s) = \int_{t=0}^{t=\infty} \mathcal{L}[H(t - \tau)f(t - \tau)](s)g(\tau) \, d\tau = \int_{t=0}^{t=\infty} \int_{\tau=0}^{\tau=\infty} e^{-st}H(t - \tau)f(t - \tau)g(\tau) \, dt \, d\tau.
\]

The definition of the Heaviside function, \(H(t - \tau) = 0\) for \(t < \tau\), \(H(t - \tau) = 1\) for \(t \geq \tau\), changes the interval of the integral

\[
F(s)G(s) = \int_{t=0}^{t=\infty} \int_{\tau=0}^{\tau=\infty} e^{-st}H(t - \tau)f(t - \tau)g(\tau) \, dt \, d\tau = \int_{t=0}^{t=\infty} \int_{\tau=0}^{\tau=\infty} e^{-st}f(t - \tau)g(\tau) \, dt \, d\tau.
\]

Now we recall the techniques studied in Calculus II (Multiple Integrals). The region defining the last double integral is shaded as in the figure 3.13, explicitly, the region is \(R = \{(t, \tau) : 0 \leq \tau \leq t\}\). We reverse the order of integration, i.e.,

\[
R = \{(t, \tau) : 0 \leq \tau, \tau \leq t\} \quad \text{(Horizontal Cut)}
\]

\[
= \{(t, \tau) : 0 \leq t, 0 \leq \tau \leq t\} \quad \text{(Vertical Cut)}.
\]

So the double integral becomes

\[
F(s)G(s) = \int_{t=0}^{t=\infty} \int_{\tau=0}^{\tau=\infty} e^{-st}f(t - \tau)g(\tau) \, dt \, d\tau \quad \text{(Horizontal Cut)}
\]
\[= \int_R e^{-st} f(t - \tau) g(\tau) dA \]
\[= \int_{t=0}^{t=\infty} \int_{\tau=0}^{\tau=t} e^{-st} f(t - \tau) g(\tau) d\tau dt \text{ (Vertical Cut)} \]
\[= \int_{t=0}^{t=\infty} \left[ \int_{\tau=0}^{\tau=t} f(t - \tau) g(\tau) d\tau \right] dt \]
\[= \int_{t=0}^{t=\infty} e^{-st} (f * g)(t) dt = \mathcal{L}[f * g](s). \]

Figure 3.13: Region defining double integral.

The inverse version of the convolution theorem is also very useful when we want to find the inverse transform of a function that is a product and we know the inverse transform of each factor.

**Theorem 3.4.3 (Inverse of Convolution Theorem).** Let \( \mathcal{L}^{-1}[F] = f \) and \( \mathcal{L}^{-1}[G] = g \). Then we have \( \mathcal{L}^{-1}[FG] = f * g \), i.e.,

\[ \mathcal{L}^{-1}[FG] = \mathcal{L}^{-1}[F] * \mathcal{L}^{-1}[G]. \]

**Example 3.4.4.** Compute \( \mathcal{L}^{-1} \left[ \frac{1}{s(s-4)^2} \right] \).

There are several ways to solve this problem. Let us see some of them.

**Answer 1. Table.** Let \( f(t) = t - \frac{1}{a} (1 - e^{-at}) \), \( a \neq 0 \). The table of the Laplace transform gives

\[ F(s) = \mathcal{L}[f](s) = \frac{a}{s^2(s+a)}. \]

So with \( a = 4 \), we observe

\[ \frac{F(s-4)}{4} = \frac{1}{s(s-4)^2}, \quad \mathcal{L}^{-1} \left[ \frac{1}{s(s-4)^2} \right] = \frac{1}{4} \mathcal{L}^{-1}[F(s-4)]. \]

The first shifting theorem implies

\[ \mathcal{L}^{-1} \left[ \frac{1}{s(s-4)^2} \right] = \frac{1}{4} \mathcal{L}^{-1}[F(s-4)] = \frac{1}{4} e^{4t} \mathcal{L}^{-1}[F](t) \]
\[ = \frac{1}{4} e^{4t} \left[ t - \frac{1}{4} (1 - e^{-4t}) \right] = \frac{1}{4} e^{4t} t - \frac{1}{16} e^{4t} + \frac{1}{16}. \]

**Answer 2. Program Mathematica.** See the figure.
ANSWER 3. PARTIAL FRACTION. We can separate the quotient as follows:

\[ \frac{1}{s(s-4)^2} = \frac{1}{16s} - \frac{1}{16s-4} + \frac{1}{4(s-4)^2}. \]

Applying the inverse Laplace transform and using the table of the Laplace transform, we get

\[ \mathcal{L}^{-1} \left[ \frac{1}{s} \right] - \frac{1}{16s} \mathcal{L}^{-1} \left[ \frac{1}{s} \right] + \frac{1}{4} \mathcal{L}^{-1} \left[ \frac{1}{s-4} \right] = \frac{1}{16} - \frac{1}{16}e^{4t} + \frac{1}{4} te^{4t}. \]

ANSWER 4. CONVOLUTION. Let us use the convolution theorem.

\[ \frac{1}{s(s-4)^2} = \frac{1}{s} \frac{1}{(s-4)^2}, \quad \mathcal{L}^{-1} \left[ \frac{1}{s} \right] \ast \mathcal{L}^{-1} \left[ \frac{1}{(s-4)^2} \right] = \mathcal{L}^{-1} \left[ \frac{1}{s} \right] \ast \mathcal{L}^{-1} \left[ \frac{1}{(s-4)^2} \right] \]

By the table of the Laplace transform, we get

\[ \mathcal{L}^{-1} \left[ \frac{1}{s(s-4)^2} \right] = \mathcal{L}^{-1} \left[ \frac{1}{s} \right] \ast \mathcal{L}^{-1} \left[ \frac{1}{(s-4)^2} \right] = 1 \ast te^{4t} = \int_0^t \tau e^{4\tau} d\tau = \frac{1}{16} - \frac{1}{16}e^{4t} + \frac{1}{4} te^{4t}. \]

Exercise 3.4.5. Use convolution to find the inverse Laplace transform of \( F(s) = \frac{2}{s^3 + 4s} \).

The convolution operation is commutative.

Theorem 3.4.6. If \( f \ast g \) is defined, then \( g \ast f \) is also defined and \( f \ast g = g \ast f \).

PROOF. The substitution \( z = t - \tau \) implies

\[ (f \ast g)(t) = \int_{\tau=0}^{\tau=t} f(t-\tau)g(\tau) d\tau = - \int_{z=0}^{z=t} f(z)g(t-z) dz = \int_{z=0}^{z=t} f(z)g(t-z) dz = (g \ast f)(t). \]

Now we use the convolution to solve an initial value problem.

Example 3.4.7. Solve the initial value problem \( y'' - 2y' - 8y = f(t), y(0) = 1 \) and \( y'(0) = 0 \). Here \( f(t) \) is any function having the Laplace transform \( F(s) \).
The inverse Laplace transform gives the solution of the given problem,
\[ \mathcal{L}[y'' - 2y' - 8y] = \mathcal{L}[f(t)], \quad \mathcal{L}[y''] - 2\mathcal{L}[y'] - 8\mathcal{L}[y] = \mathcal{L}[f], \]
\[ s^2Y - sy(0) - y'(0) - 2(sY - y(0)) - 8Y = F, \quad (s^2 - 2s - 8) Y = F + s - 2, \]
\[ Y = \frac{1}{s^2 - 2s - 8} F + \frac{s - 2}{s^2 - 2s - 8} = \frac{1}{(s + 2)(s - 4)} F + \frac{s - 2}{(s + 2)(s - 4)}. \]

The partial fraction technique implies
\[ Y = \frac{1}{(s + 2)(s - 4)} F + \frac{s - 2}{(s + 2)(s - 4)}. \]
\[ = \frac{1}{6} \frac{1}{s - 4} F + \frac{1}{6} \frac{1}{s + 4} F + \frac{1}{3} \frac{1}{s - 4} + \frac{2}{3} \frac{1}{s + 4}. \]

The inverse Laplace transform gives the solution of the given problem,
\[ y(t) = \mathcal{L}^{-1}[Y] = \frac{1}{6} \mathcal{L}^{-1}\left[ \frac{1}{s - 4} F - \frac{1}{6} \mathcal{L}^{-1}\left[ \frac{1}{s + 4} F \right] + \frac{1}{3} \mathcal{L}^{-1}\left[ \frac{1}{s - 4} \right] + \frac{2}{3} \mathcal{L}^{-1}\left[ \frac{1}{s + 4} \right] \right] \]
\[ = \frac{1}{6} \mathcal{L}^{-1}\left[ \frac{1}{s - 4} \right] \ast \mathcal{L}^{-1}[F] - \frac{1}{6} \mathcal{L}^{-1}\left[ \frac{1}{s + 4} \right] \ast \mathcal{L}^{-1}[F] + \frac{1}{3} \mathcal{L}^{-1}\left[ \frac{1}{s - 4} \right] + \frac{2}{3} \mathcal{L}^{-1}\left[ \frac{1}{s + 4} \right] \]
\[ = \frac{1}{6} e^{4t} * f(t) - \frac{1}{6} e^{-2t} * f(t) + \frac{1}{3} e^{4t} + \frac{2}{3} e^{-2t}. \qed \]

We end this section with another application of the convolution to the integral equations.

**Example 3.4.8.** Solve the integral equation,
\[ f(t) = 2t^2 + \int_0^t f(t - \tau)e^{-\tau} d\tau. \]

That is, find a function \( f(t) \) satisfying the given equation.

**Answer 1. Convolution.** We observe the integral is the convolution of \( f \) with \( e^{-t} \),
\[ \int_0^t f(t - \tau)e^{-\tau} d\tau = f(t) \ast e^{-t}, \]
and so the given equation becomes
\[ f(t) = 4t + f(t) \ast e^{-t}. \]

Applying the Laplace transform, we get
\[ \mathcal{L}[f] = 4\mathcal{L}[t] + \mathcal{L}[f(t) \ast e^{-t}], \quad \mathcal{L}[f] = 4 \frac{1}{s^3} + \mathcal{L}[f] \mathcal{L}[e^{-t}], \quad F = 4 \frac{1}{s^3} + F \frac{1}{s + 1}, \quad F(s) = 4 \frac{1}{s^3} + \frac{4}{s^4}. \]

The inverse Laplace transform implies
\[ f(t) = \mathcal{L}^{-1}[F] = 4 \mathcal{L}^{-1}\left[ \frac{1}{s^3} \right] + 4 \mathcal{L}^{-1}\left[ \frac{1}{s^4} \right] = 4 \frac{t^2}{2} + 4 \frac{t^3}{6} = 2t^2 + \frac{2}{3} t^3. \qed \]

**Answer 2. Differential Equation.** First, let us change the integral in the given equation with the substitution \( z = t - \tau \),
\[ \int_0^t f(t - \tau)e^{-\tau} d\tau = - \int_z^t f(z)e^{z-t} dz = e^{-t} \int_{z=0}^{z=t} f(z)e^z dz. \]
Hence, the given equation becomes
\[ f(t) = 2t^2 + e^{-t} \int_0^t f(z) e^{z} \, dz. \]

We will find a differential equation with the initial condition. It is easy to see
\[ f(0) = 2(0) + e^{-0} \int_0^0 f(z) e^{z} \, dz = 0, \quad i.e., \quad f(0) = 0. \]

Now we differentiate the whole equation with respect to \( t \). Then we get
\[ f'(t) = 4t - e^{-t} \int_0^t f(z) e^{z} \, dz + e^{-t} f(t) e^{t} = 4t + f(t) - e^{-t} \int_0^t f(z) e^{z} \, dz, \]
\[ e^{-t} \int_0^t f(z) e^{z} \, dz = 4t + f(t) - f'(t). \]

The left–hand side of the last equation is also in the given problem. Combining two equations,
\[ f(t) - 2t^2 = e^{-t} \int_0^t f(z) e^{z} \, dz = 4t + f(t) - f'(t), \quad f'(t) = 4t + 2t^2, \]
which is a differential equation with \( f(0) = 0 \). Just integrating it, we deduce
\[ f(t) = 2t^2 + \frac{2}{3} t^3, \]
which is exactly same as the one in the ANSWER 1.

\[ \square \]

**Remark 3.4.9 (ASIDE).** Let \( I \) be the identity function, i.e., \( I(t) = t \). For the composition of functions, we recall that \( f \circ I = f = I \circ f \). We can develop the similar argument on the convolution of functions. Is there (sort of) a function \( g \) satisfying \( f * g = f = g * f \)? The answer is . . . . . . Yes, it is called the (Dirac) Delta Distribution, \( \delta(t) \), i.e., \( f * \delta = f = \delta * f \). On this topic, you may study the next Section 3.5 Unit Impulses and the Dirac Delta Function.

§3.5 Unit Impulses and the Dirac Delta Function.
Skip. Please read the textbook.

§3.6 Laplace Transform Solution of Systems.
Skip. Please read the textbook.

§3.7 Differential Equations with Polynomial Coefficients.
Skip. Please read the textbook.