Solving an Integro-Differential Equation by Legendre Polynomial and Block-Pulse Functions

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Abstract

In this paper, hybrid Legendre Block-Pulse functions are developed to find an approximate solution for an integro-differential equation. Hybrid Legendre Block-Pulse functions are developed by combining Block-Pulse functions on [0, 1] and Legendre polynomials. By using this method integro-differential equations reduce to a system of linear equations.

Key words: Block-Pulse functions, Volterra and Fredholm integral equation, Integro-differential equation.

1 Introduction

In this paper, we will use a simple basis for solving an integro-differential equation. This basis is a combination of Block-Pulse functions on [0, 1], and Legendre polynomials, that is called the hybrid Legendre Block-Pulse functions.

1.1 Definition

Consider the Legendre polynomials \( p_m(t) \) on the interval \([-1, 1]\):

\[
\begin{align*}
p_0(t) &= 1, \quad p_1(t) = t, \ldots, \\
p_{m+1}(t) &= \frac{2m+1}{m+1} + p_m(t) - \frac{m}{m+1}p_{m-1}(t), \quad m = 1, 2, \ldots
\end{align*}
\]

The set \( \{p_m(t); \ m = 0, 1, \ldots \} \) in the Hilbert space \( L^2[-1, 1] \) is a complete orthogonal set on \([1, 2] \).
1.2 Lemma

Let \( x(t) \in H^k(-1,1) \) (a Sobolev space) and let \( x_j(t) = \sum_{i=0}^j a_i L_i(t) \) be the best approximation polynomial of \( x(t) \) in the \( L^2 \)-norm, then

\[
\|x(t) - x_j(t)\|_{L^2[-1,1]} \leq c_0 j^{-k} \|x(t)\|_{H^k(-1,1)}
\]

where \( c_0 \) is a positive constant, which depends on the selected norm and is independent of \( x(t), j \) (see [3]).

1.3 Definition

A set of Block-Pulse functions \( b_i(\lambda), i = 1, 2, \ldots, m \), on the interval \([0,1]\) are defined as follows:

\[
b_i(\lambda) = \begin{cases} 
1, & \frac{i-1}{m} \leq \lambda < \frac{i}{m}; \\
0, & \text{otherwise}.
\end{cases}
\] (1.2)

The Block-Pulse functions on \([0,1]\) are disjoint, that is, for \( i = 1, 2, \ldots, m \), they satisfy an orthogonality property on \([0,1]\).

1.4 Definition

For \( m = 0, 1, 2, \ldots, M-1 \) and \( n = 1, 2, \ldots, N \) the hybrid Legendre Block-Pulse functions are defined as:

\[
b(n,m,t) = \begin{cases} 
P_m(2Nt - 2n + 1), & \frac{n-1}{N} \leq t < \frac{n}{N}; \\
0, & \text{otherwise}.
\end{cases}
\] (1.3)

1.5 The operational matrix

If

\[
B(t) = [b(1,0,t), b(1,1,t), \ldots, b(1,M-1,t), b(2,0,t), \ldots, b(N,M-1,t)]^T
\]

is a vector function of hybrid Legendre Block-Pulse functions on \([0,1]\), the integration of the vector \( B(t) \) can be obtained as:

\[
\int_0^1 B(t') \, dt' \simeq PB(t)
\] (1.4)

where \( P \) is an \( MN \times MN \) matrix, that is called the operation matrix for hybrid Legendre Block-Pulse functions. Then the operation matrix \( P \) has the following
form [4, 5]

\[
P = \begin{pmatrix}
    E & H & \ldots & H \\
    0 & E & \ldots & H \\
    \vdots & \vdots & \ddots & \vdots \\
    0 & 0 & \ldots & E
\end{pmatrix}
\]  

(1.5)

where \( H \) is an \( M \times M \) matrix and is defined as follows:

\[
H = \frac{1}{N} \begin{pmatrix}
    1 & 0 & \ldots & 0 \\
    0 & 0 & \ldots & 0 \\
    \vdots & \vdots & \ddots & \vdots \\
    0 & 0 & \ldots & 0
\end{pmatrix};
\]

(1.6)

also \( E \) is an \( M \times M \) matrix on the interval \([0, \frac{1}{N}]\) and is defined as follows [5, 6]:

\[
E = \frac{1}{2N} \begin{pmatrix}
    1 & 1 & 0 & 0 & \ldots & 0 & 0 & 0 \\
    \frac{1}{2} & 0 & \frac{1}{3} & 0 & \ldots & 0 & 0 & 0 \\
    0 & -\frac{1}{2} & 0 & -\frac{1}{3} & \ldots & 0 & 0 & 0 \\
    \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
    0 & 0 & 0 & 0 & \ldots & -\frac{1}{2M-3} & 0 & \frac{1}{2M-3} \\
    0 & 0 & 0 & 0 & \ldots & 0 & -\frac{1}{2M-1} & 0
\end{pmatrix}
\]

(1.7)

2 Function approximation

A function \( x(t) \in L^2[0, 1] \) may be expanded as

\[
x(t) = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} X(n, m)b(n, m, t),
\]

(2.1)

where

\[
X(n, m) = \frac{(x(t), b(n, m, t))}{(b(n, m, t), b(n, m, t))}
\]

(2.2)

where \((\cdot, \cdot)\) denotes the inner product. If the infinite series in (2.1) is truncated, then (2.1) can be written as

\[
x(t) \simeq X_{NM}(t) = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} X(n, m)b(n, m, t) = X^T B(t),
\]

(2.3)
Solving integro-differential equation by Legendre polynomial

where $B(t)$ is a vector function and $X$ is given by

$$X = [X(1,0), X(1,1), \ldots, X(1,M-1), X(2,0), \ldots, X(N,M-1)]^T.$$

We can also approximate the function $k(t,s) \in L^2([0,1] \times [0,1])$ as follows:

$$k(t,s) \simeq k_{NM}(t,s) = B^T(t)kB(s),$$ (2.4)

where $k$ is an $MN \times MN$ matrix such that

$$k_{ij} = \frac{(B_i(t), (k(t,s), B_j(s)))}{(B_i(t), B_i(t))(B_j(s), B_j(s))}, \quad i,j = 1, 2, \ldots, MN.$$ (2.5)

We also define the matrix $D$ as follows:

$$D = \int_0^1 B(t)B^T(t) dt.$$ (2.6)

For the hybrid Legendre Block-Pulse functions, $D$ has the following form:

$$D = \begin{pmatrix}
D_1 & 0 & \ldots & 0 \\
0 & D_2 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & D_N
\end{pmatrix},$$ (2.7)

where $D_i$ is defined as follows:

$$D_i = \frac{1}{N} \int T(t)T^T(t) dt.$$

3 Integro-differential equation

Consider the following integro-differential equation:

$$q(t)y'(t) = \int_0^1 k(t,s)y(s) ds + r(t)y(t) + x(t),$$

$$y(0) = y_0,$$ (3.1)

where $x, q, r \in L^2[0,1], k \in L^2([0,1] \times [0,1])$ and $y$ is an unknown function [7]. If we approximate $x, q, r, y'$ and $k$ by (2.1)–(2.4) as follows:

$$x(t) \simeq X^T B(t), \quad y(t) \simeq Y^T B(t), \quad k(t,s) \simeq B^T(t)KB(s),$$
then
\[
y(t) = \int_0^t y'(t') \, dt' + y(0)
\]
\[
\simeq \int_0^t Y^T B(t') \, dt' + Y_0^T B(t)
\]
\[
\simeq Y^T DB(t) + Y_0^T B(t)
\]
\[
= (Y^{TT} + Y_0^T) B(t).
\]

With substituting in (3.1) we have
\[
Y^T = Y^T D + Y_0^T \Rightarrow y(t) \simeq Y^T B(t).
\]

4 Numerical experiments

Example 1. Consider the equation with exact solution \( y(t) = e^t \):
\[
y'(t) = \int_0^t -0^1 e^{s}\, y(s) \, ds + y(t) + \frac{1 - e^{t+1}}{t+1},
\]
\[
y(0) = 1.
\]
The solution for \( y(t) \) is obtained by the method of Section 3. Results are shown in Table 1.

Example 2. Consider the equation with exact solution \( y(t) = \cos(2\pi t) \):
\[
y'(t) = \int_0^1 \sin(4\pi t + 2\pi s) y(s) \, ds + y(t) - \cos(2\pi t) - 2\pi \sin(2\pi t) - \frac{1}{2} \sin(4\pi t),
\]
\[
y(0) = 1.
\]
The solution for \( y(t) \) is obtained by the method of Section 3. Results are shown in Table 2.

<table>
<thead>
<tr>
<th>Table 1: Results for Example 1</th>
<th>Table 2: Results for Example 2</th>
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</table>
5 Conclusion

If we solve the integro-differential equation using orthogonal continuous or piecewise constant functions, the accuracy of the method will be worse. Whereas, using hybrid Legendre and Block-Pulse functions the accuracy of system will improve using suitable $M$ and $N$ because the hybrid Legendre and Block-Pulse functions are orthogonal piecewise continuous functions and have high flexibility.

References


