Blow up of Solutions for a Class of Nonlinear Wave Equations

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Abstract
In this work, we study the blow up of solutions to the initial boundary value problem for a class of nonlinear wave equations with a dissipative term.

1 Introduction
In this work, we study the blow up of solutions of initial boundary value problem for a class of nonlinear wave equations with a dissipative term:

\[ u_{tt} = \text{div} \sigma(\nabla u) + \Delta u_t - \Delta^2 u \quad \text{in} \quad \Omega \times (0, +\infty), \]

\[ u|_{\partial \Omega} = 0, \quad \frac{\partial u}{\partial \nu}|_{\partial \Omega} = 0 \quad \text{on} \quad (0, +\infty), \]

\[ u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), \quad x \in \Omega, \]

where $\Omega$ is a bounded domain in $\mathbb{R}^n$ with a sufficiently smooth boundary $\partial \Omega$, $\nu$ is the outward normal to the boundary and $\sigma(s)$ are given nonlinear functions.

The study of nonlinear evolution equations with linear damping or dissipative term has been considered by many authors; see [1]–[7]. In our study, we establish a blow up result for solutions with negative energy. The proof of our technique is similar to the one in [7].

2 Blow up of solution

For this purpose, we define

\[ E(t) = \frac{1}{2}\|u_t(t)\|^2 + \frac{1}{2}\|\Delta u(t)\|^2 + \int_0^t \int_{\Omega} \sigma(s) ds \, dx, \quad t \geq 0, \]

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where
\[ \sigma(s) \in C(\mathbb{R}), \ s \in \mathbb{R}, \ \sigma(s)s \leq k \int_0^s \sigma(\tau) d\tau \leq -k\beta|s|^{m+1}, \]

\( k > 2 \) and \( \beta > 0 \) are constants, also \( 1 < m \leq 3 \).

**Theorem 1** Let \( u \) be the solution of problem (1) – (3). Assume that the following conditions are valid:

\[ u_0 \in H_0^2(\Omega), \ u_1 \in L_2(\Omega), \]

\[ E(0) = \frac{1}{2} u_1^2 + \frac{1}{2} \Delta u_0^2 + \int_0^{\tau_0} \sigma(s) ds \ dx < 0. \]

Then the solution \( u \) blows up in finite time

\[ T \leq \begin{cases} \left( \frac{3-m}{t_1} + \frac{3-m}{C_8(\alpha-1)y^{\alpha-1}(t_1)} \right)^{\frac{2}{\alpha-1}}, & m < 3, \\ t_1 \cdot \exp \left( \frac{1}{C_8(\alpha-1)y^{\alpha-1}(t_1)} \right), & m = 3, \end{cases} \]

where \( t_1 \) and \( y \) will be defined respectively by (17) and (18), \( C_8 \) and \( \alpha > 1 \) are constants to be defined later.

**Proof.** By multiplying equation (1) by \( u_t \) and integrating the new equation over \( \Omega \), we obtain

\[ E'(t) + \| \nabla u_t(t) \|^2 = 0, \]

\[ E(t) \leq E(0) < 0, \ t \geq 0. \]

Let

\[ F(t) = \| u(t) \|^2 + \int_0^t \| \nabla u(\tau) \|^2 d\tau, \]

then

\[ F'(t) = 2(u, u_t) + \| \nabla u(t) \|^2; \]

\[ F''(t) = 2 \left( \| u_t(t) \|^2_2 - \| \Delta u(t) \|^2_2 - \int_{\Omega} \sigma(u_x) u_{xx} \ dx \right) \]

\[ \geq 2 \left( \| u_t(t) \|^2_2 - \| \Delta u(t) \|^2_2 - k \int_0^{\tau_0} \int_{\Omega} \sigma(s) ds \ dx \right) \]

\[ \geq 2 \left( 2\| u_t(t) \|^2_2 - (k - 2) \int_0^{\tau_0} \int_{\Omega} \sigma(s) ds \ dx - 2E(0) \right) \]

\[ \geq 2 \left( 2\| u_t(t) \|^2_2 + (k - 2)\beta \| \nabla u(t) \|_m^{m+1} - 2E(0) \right), \ t > 0. \]
where the assumption (5) and the fact that
\[ k \int_0^{u_x} \sigma(s) \, ds \, dx \leq 2E(0) - \|u_t(t)\|_2^2 + \|\Delta u(t)\|_2^2 + (k - 2) \int_0^{u_x} \sigma(s) \, ds \, dx \]
have been used. Taking the inequality (10) and integrating this, we obtain
\[ F'(t) \geq 2(k - 2)\beta \int_0^t \|\nabla u(\tau)\|_{m+1}^{m+1} \, d\tau - 4E(0)t + F'(0), \quad t > 0. \tag{11} \]
After this calculation, we could add the inequalities (10) with (11), then we get
\[ F''(t) + F'(t) \geq 4\|u_t(t)\|_2^2 + 2(k - 2)\beta \left( \|\nabla u(t)\|_{m+1}^{m+1} + \int_0^t \|\nabla u(\tau)\|_{m+1}^{m+1} \, d\tau \right) - 4E(0)(1 + t) + F'(0) = g(t), \quad t > 0. \tag{12} \]
Take \( p = \frac{m+3}{m+1} \), obviously \( 2 < p < m+1 \) and \( p' = \frac{m+3}{m+1} \) \((< 2) \). By using the Young inequality and the Sobolev-Poincaré inequality,
\[
|\langle u, u_t \rangle| \leq \frac{1}{p} \|u(t)\|_p^p + \frac{1}{p'} \|u_t(t)\|_{p'}^{p'} \leq C_1 \left[ (\|\nabla u(t)\|_{m+1}^{m+1})^\mu + (\|u_t(t)\|_2^2)^\mu \right], \quad t > 0, (13)
\]
where in this inequality and in the sequel \( C_i \) \((i = 1, 2, \ldots) \) denote positive constants independent of \( t \), \( \mu = \frac{m+3}{2(m+1)} \) \((< 1) \). By the Sobolev-Poincaré inequality and the Hölder inequality
\[
\|\nabla u(t)\|_{m+1}^{m+1} \geq C_3 \left( \|u(t)\|_2^2 \right)^{\frac{m+1}{m+1}}, \quad t > 0, \tag{14}
\]
\[
\int_0^t \|\nabla u(\tau)\|_{m+1}^{m+1} \, d\tau \geq C_4 t^{\frac{1-m}{2}} \left( \int_0^t \|\nabla u(\tau)\|_2^2 \, d\tau \right)^{\frac{m+1}{2}}. \tag{15}
\]
By using the inequalities (13)–(15), we obtain
\[
g(t) \geq C_5 \left( 3\|\nabla u(t)\|_{m+1}^{m+1} + \|u_t(t)\|_2^2 + \int_0^t \|\nabla u(\tau)\|_{m+1}^{m+1} \, d\tau \right) - 4E(0)t + F'(0)\geq C_6 \left( (\langle u, u_t \rangle)^{1/\mu} + (\|u(t)\|_2^2)^{\frac{m+1}{2}} + (\|\nabla u(t)\|_2^2)^{\frac{m+1}{2}} + t^{1-m/2} \left( \int_0^t \|\nabla u(\tau)\|_2^2 \, d\tau \right)^{\frac{m+1}{2}} \right) - 4E(0)t + F'(0)\]
\[ \geq C t^{\frac{1-m}{\alpha}} \left( (u, u_t)^{\alpha} + \left( \|u(t)\|^2 \right)^{\alpha} + \left( \|\nabla u(t)\|^2 \right)^{\alpha} + \left( \int_0^t \|\nabla u(\tau)\|^2 d\tau \right)^{\alpha} \right) \]

\[-4E(0)t + F'(0) - C t^{\frac{1-m}{2}}, \quad t \geq 1, \quad (16)\]

where in this inequality and in the sequel \( \alpha = \frac{1}{\mu} > 1 \). Since \(-4E(0)t + F'(0) - C t^{\frac{1-m}{2}} \to \infty \) as \( t \to \infty \), there must be a \( t_1 \geq 1 \) such that

\[-4E(0)t + F'(0) - C t^{\frac{1-m}{2}} \geq 0 \quad \text{as} \quad t \geq t_1. \quad (17)\]

Let

\[ y(t) = F'(t) + F(t), \quad (18) \]

then from the inequality (11) and the equality (8) we obtain \( y(t) > 0 \) as \( t \geq t_1 \). By using the inequality

\[ (a_1 + \cdots + a_\ell)^n \leq 2^{n-1}(\ell-1) (a_1^n + \cdots + a_\ell^n), \]

where \( a_i \geq 0 \) \( (i = 1, \ldots, \ell) \) and \( n > 1 \) are real numbers, by virtue of (17) and using the inequality (16), we get

\[ g(t) \geq C t^{\frac{1-m}{2}} y^n(t), \quad t \geq t_1. \quad (19) \]

So combining (12) with (19) gives

\[ y'(t) \geq C s t^{\frac{1-m}{2}} y^n(t), \quad t \geq t_1. \quad (20) \]

Therefore, there exists a positive constant

\[ T = \begin{cases} \left[ t_1^{\frac{3-m}{2}} + \frac{3-m}{2} \left( \frac{3-m}{\alpha-1} y^{\alpha-1}(t_1) \right) \right]^{\frac{2}{3-m}}, & m < 3, \\ t_1 \cdot \exp \left( \frac{1}{\alpha-1} y^{\alpha-1}(t_1) \right), & m = 3, \end{cases} \quad (21) \]

such that

\[ y(t) \to \infty \quad \text{as} \quad t \to T^-. \quad (22) \]

By using (8), (9) and (22), we obtain

\[ 2\|u(t)\|^2 + \|u_t(t)\|^2 + \|\nabla u(t)\|^2 + \int_0^t \|\nabla u(\tau)\|^2 d\tau \geq F'(t) + F(t) \to \infty \quad \text{as} \quad t \to T^- \quad (23) \]

So (23) implies

\[ \|u(t)\|^2 + \|u_t(t)\|^2 + \|\nabla u(t)\|^2 + \int_0^t \|\nabla u(\tau)\|^2 d\tau \to \infty \quad \text{as} \quad t \to T^- \]

This completes the proof.
References


