On the Approximation of Singular Integrals of Cauchy Type

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Abstract

The aim of this work is to approximate numerically the singular integral of Cauchy type on a piecewise smooth curve by expressions based on the cubic spline, which is one of the recent ideas in numerical analysis.

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Let $\Gamma$ be a a piecewise regular curve, in other words, $\Gamma$ consists of a finite number of smooth non-intersecting contours in a complex plane, where $\Gamma$ can be represented in the form

$$t(s) = x(s) + iy(s), \quad a \leq s \leq b, \quad a, b \in \mathbb{R},$$

where $x(s)$ and $y(s)$ are continuous functions in the interval $[a, b]$ with the following property:

The functions $x(s)$ and $y(s)$ have continuous first derivatives $x'(s)$ and $y'(s)$ within the interval $[a, b]$, including the endpoints, and these derivatives are never simultaneously zero.

Let $F(t_0)$ be a singular integral defined by

$$F(t_0) = \frac{1}{\pi i} \int_{\Gamma} \frac{\varphi(t)}{t - t_0} dt, \quad t_0 \in \Gamma. \quad (1)$$

For the existence of the principal value for a given density $\varphi(t)$, we will need more than mere continuity, in other words, the density $\varphi(t)$ has to satisfy the Hölder condition ($\varphi \in H(\mu)$) [2].
Let us now consider an arbitrary natural number $N$, generally we take it large enough, we divide the interval $[a, b]$ into $N$ subintervals of $[a, b] = \{ a = s_0 < s_1 < \cdots < s_N = b \}$ called $I_i$ to $I_N$, so that $I_{\sigma+1} = [s_\sigma, s_{\sigma+1}]$. Also define $h_{\sigma+1} = s_{\sigma+1} - s_\sigma$, noting that the subintervals need not be of equal length.

But, in our case and for reasons of programming one takes the subintervals of the same length, into $N$ equal parts by the points

$$ s_\sigma = a + \sigma \frac{l}{N}, \quad l = b - a, \quad \sigma = 0, 1, 2, \ldots, N. $$

Denoting $t_\sigma = t(s_\sigma)$ and using the smoothness of $\Gamma$, we can take $h_{\sigma+1} = t_{\sigma+1} - t_\sigma$ [3, 7] and assuming that $\sigma, \nu = 0, 1, 2, \ldots, N - 1$, we consider now that the point $t_0$ belongs to the arc $t_{\nu}t_{\nu+1}$, where $t_{\nu}$ denotes the smallest arc with ends $t_\nu$ and $t_{\nu+1}$ [3, 6].

For the arbitrary numbers $\sigma, \nu$ from 1, 2, $\ldots$, $N - 1$ we define the function $\beta_{\sigma\nu}(\varphi; t, t_0)$ dependent on $\varphi$, $t$ and $t_0$ by

$$ \beta_{\sigma\nu}(\varphi; t, t_0) = (S_3(\varphi; t, \sigma) - S_3(\varphi; t_0, \nu)) \frac{2(t - t_0)}{(t_\sigma - t_0) + (t_{\sigma+1} - t_0)}, \quad (2) $$

where the expression $S_3(\varphi; t, \sigma)$ denotes the cubic spline to the function density $\varphi(t)$ on the curve $\Gamma$ given by the following formula

$$ S_3(\varphi; t, \sigma) = \frac{M_\sigma(t_{\sigma+1} - t)^3}{6h_{\sigma+1}} + \frac{M_{\sigma+1}(t - t_\sigma)^3}{6h_{\sigma+1}} $$

$$ + \left( \varphi(t_\sigma) - \frac{M_\sigma h_{\sigma+1}^2}{6} \right) \frac{t_{\sigma+1} - t}{h_{\sigma+1}} $$

$$ + \left( \varphi(t_{\sigma+1}) - \frac{M_{\sigma+1} h_{\sigma+1}^2}{6} \right) \frac{t - t_\sigma}{h_{\sigma+1}} $$

and the density $\varphi$ still represents a given function on the curve $\Gamma$ of class $H(\mu)$.

Seeing that the equality $[(t_\sigma - t_0) + (t_{\sigma+1} - t_0)] = 0$ is possible only when $\sigma = \nu$, in this case we take the function $\beta_{\sigma\sigma}(\varphi; t, t_0)$ omitting the expression $\frac{2(t - t_0)}{(t_\sigma - t_0) + (t_{\sigma+1} - t_0)}$, as given by

$$ \beta_{\sigma\sigma}(\varphi; t, t_0) = S_3(\varphi; t, \sigma) - S_3(\varphi; t_0, \sigma). \quad (3) $$

It is simple to see that, for $N$ large enough, the limit of the expression $\frac{2(t - t_0)}{(t_\sigma - t_0) + (t_{\sigma+1} - t_0)}$ is equal to the unit. However, the expressions (2) and (3) are almost equal, so we can confirm that the function $\beta_{\sigma\nu}(\varphi; t, t_0)$ is defined for all values of the variables $t, t_0 \in \Gamma$, and almost continuous at all points, for all $\sigma, \nu = 0, 1, \ldots, N - 1$.

Now we define the function

$$ \psi_{\sigma\nu}(\varphi; t, t_0) = \left\{ \begin{array}{ll} \varphi(t_\sigma) + \beta_{\sigma\nu}(\varphi; t, t_0), & t \in \tau_\sigma \tau_{\sigma+1}, \; t_0 \in \tau_{\nu} \tau_{\nu+1}, \\
\sigma = 0, 1, \ldots, N - 1; & \nu = 0, 1, \ldots, N - 1. \end{array} \right. $$
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It can be easily seen that the function $\beta_{\sigma \nu}(\varphi; t, t_0)\) contains $(t - t_0)$ as a factor, for all $\sigma, \nu = 0, 1, \ldots, N-1$, whence, the function $\psi_{\sigma \nu}(\varphi; t, t_0)$ admits the following representation

$$
\psi_{\sigma \nu}(\varphi; t, t_0) = \varphi(t_0) + (t - t_0)Q_{\sigma \nu}(\varphi; t, t_0).
$$

(4)

After this construction, one replaces the singular integral

$$
F(t_0) = \frac{1}{\pi i} \int_{\Gamma} \frac{\varphi(t)}{t - t_0}\, dt
$$

by the following ones

$$
S(\varphi; t_0) = \frac{1}{\pi i} \int_{\Gamma} \frac{\psi_{\sigma \nu}(\varphi; t, t_0)}{t - t_0}\, dt = \varphi(t_0) + \frac{1}{\pi i} \int_{\Gamma} Q_{\sigma \nu}(\varphi; t, t_0)\, dt.
$$

(5)

Let us now cite the theorem concerning the accuracy of approximation of singular integrals (1) by expressions of the form (5).

**Theorem** Let $\Gamma$ be a rectifiable simple path of finite length and let $\varphi$ be a density satisfying the Hölder condition $(H(\mu))$, then the following estimation

$$
| F(t_0) - S(\varphi; t_0) | \leq \frac{C_N}{N^\mu}, \quad N > 1,
$$

holds, where the constant $C_N$ depends only of the curve $\Gamma$. Furthermore, if we suppose that $\varphi$ and its first derivatives are continuous and

$$
\max_{t \in \Gamma} | \varphi^{(4)}(t) | = M,
$$

then one has

$$
| F(t_0) - S(\varphi; t_0) | \leq \frac{C_N}{N^{\mu+4}}, \quad N > 1.
$$

For the sake of simplicity, we try to prove only the first estimate. Indeed, for $t \in t_\sigma t_{\sigma + 1}$ and $t_0 \in t_\nu t_{\nu + 1}$, we consider

$$
\varphi(t) - \psi_{\sigma \nu}(\varphi; t, t_0) = \varphi(t) - \{ \varphi(t_0) + \beta_{\sigma \nu}(\varphi; t, t_0) \}.
$$

For the sake of simplicity we take the cubic spline as a polynomial of degree three characterized by its moments $M_{\sigma}$,

$$
S_3(\varphi; t, \sigma) = \alpha_{\sigma} + \beta_{\sigma}(t - t_\sigma) + \gamma_{\sigma}(t - t_\sigma)^2 + \delta_{\sigma}(t - t_\sigma)^3 \quad \text{for } t \in [t_\sigma, t_{\sigma + 1}],
$$

where

$$
\alpha_{\sigma} = \varphi(t_\sigma),
\beta_{\sigma} = \frac{\varphi(t_{\sigma + 1}) - \varphi(t_\sigma)}{h_{\sigma + 1}} = \frac{2M_{\sigma} + M_{\sigma + 1}}{6}h_{\sigma + 1},
\gamma_{\sigma} = \frac{M_{\sigma}}{2},
\delta_{\sigma} = \frac{M_{\sigma + 1} - M_{\sigma}}{6h_{\sigma + 1}}.
$$
For all \( t \in t_{\sigma}t_{\sigma + 1} \) and \( t_0 \in t_{\nu}t_{\nu + 1}, \sigma \neq \nu \), we can write
\[
\varphi(t) - \psi_{\sigma\nu}(\varphi; t, t_0) = \varphi(t) - \varphi(t_0) \\
- \{ \varphi(t_{\sigma}) + \beta_{\sigma}(t - t_{\sigma}) + \gamma_{\sigma}(t - t_{\sigma})^2 + \delta_{\sigma}(t - t_{\sigma})^3 \\
- \varphi(t_{\nu}) - \beta_{\nu}(t - t_{\nu}) - \gamma_{\nu}(t - t_{\nu})^2 \\
- \delta_{\nu}(t - t_{\nu})^3 \} \frac{2(t - t_0)}{(t_{\sigma} - t_0) + (t_{\sigma + 1} - t_0)}
\] (6)

If \( \sigma = \nu \), we can easily put our expression in the form
\[
\varphi(t) - \psi_{\sigma\nu}(\varphi; t, t_0) = \varphi(t) - \varphi(t_0) \\
- \{ \beta_{\sigma} + \gamma_{\sigma}((t - t_{\sigma}) + (t_0 - t_{\sigma})) + \delta_{\sigma}((t - t_{\sigma})^2 \\
+ (t - t_{\sigma})(t_0 - t_{\sigma}) + (t_0 - t_{\sigma})^2 \} (t - t_0).
\] (7)

Taking into account the expressions (6), (7) above, we have
\[
\frac{1}{\pi i} \int_{\Gamma} \frac{\varphi(t) - \psi_{\sigma\nu}(\varphi; t, t_0)}{t - t_0} dt = \sum_{\sigma = 0}^{N-1} \frac{1}{\pi i} \int_{\Gamma} \left\{ \frac{\varphi(t) - \varphi(t_0)}{t - t_0} \\
- \left[ \varphi(t_{\sigma}) + \beta_{\sigma}(t - t_{\sigma}) + \gamma_{\sigma}(t - t_{\sigma})^2 + \delta_{\sigma}(t - t_{\sigma})^3 \\
- \varphi(t_{\nu}) - \beta_{\nu}(t - t_{\nu}) - \gamma_{\nu}(t - t_{\nu})^2 \\
- \delta_{\nu}(t - t_{\nu})^3 \right] \frac{1}{t_{\sigma} + t_{\sigma + 1} - t_0} \right\} dt
\] (8)

Passing now to the estimation of the expression (8), we have for \( t_0 \in t_{\nu}t_{\nu + 1} \) and \( \sigma \neq \nu \) the relation
\[
\left| \sum_{\sigma = 0}^{N-1} \int_{t_{\sigma}t_{\sigma + 1}} \left\{ \frac{\varphi(t) - \varphi(t_0)}{t - t_0} - \left[ \varphi(t_{\sigma}) - \varphi(t_0) + \beta_{\sigma}(t - t_{\sigma}) \right. \\
- \beta_{\nu}(t - t_{\nu}) \left. \right] \frac{1}{t_{\sigma} + t_{\sigma + 1} - t_0} \right\} dt \right| = O(N^{-\mu}).
\]

Naturally, this estimation given above is obtained using expressions of \( \beta_{\sigma} \) and \( \varphi \in H(\mu) \) [2]. Besides, it is easy to see that
\[
\left| \sum_{\sigma = 0}^{N-1} \int_{t_{\sigma}t_{\sigma + 1}} \left\{ \gamma_{\sigma}(t - t_{\sigma})^2 - \gamma_{\nu}(t_0 - t_{\nu})^2 \right\} \frac{1}{t_{\sigma} + t_{\sigma + 1} - t_0} dt \right| = O(N^{-2})
\]
and
\[ \left| \sum_{\sigma=0}^{N-1} \int_{t_{\sigma}t_{\sigma+1}} \{ \delta_\sigma (t - t_\sigma)^3 - \delta_\nu (t_0 - t_\nu)^3 \} \frac{1}{t_{\sigma} + t_{\sigma+1} - t_0} \, dt \right| = O(N^{-2}). \]

Further, using again the condition \( \varphi \in H(\mu) \) and the condition of smoothness of \( \Gamma \), we have
\[ \left| \int_{t_\nu t_{\nu+1}} \frac{\varphi(t) - \varphi(t_0)}{t - t_0} \, dt \right| \leq A \int_{s_\nu}^{s_{\nu+1}} |s - s_0|^{\mu-1} \, ds = O(N^{-\mu}). \]

And again on the base of \( \varphi \in H(\mu) \) for the expression of \( \beta_\nu \), we can easily come to
\[ \left| \int_{t_\nu t_{\nu+1}} \left\{ \beta_\nu + \gamma_\nu ((t - t_\nu) + (t_0 - t_\nu)) + \delta_\nu ((t - t_\nu)^2 + (t - t_\nu)(t_0 - t_\nu) + (t_0 - t_\nu)^2) \right\} \, dt \right| = O(N^{-\mu}). \]

**Numerical experiments:** Using our approximation, we apply the algorithms to singular integrals and we present results concerning the accuracy of the calculations. In these numerical experiments each table \( I \) represents the exact value of the singular integral and \( \bar{I} \) corresponds to the approximate calculation produced by our approximation at points of interpolation.

**Example** Consider the singular integral
\[ I = F(t_0) = \frac{1}{\pi i} \int_\Gamma \frac{\varphi(t)}{t - t_0} \, dt, \]
where the curve \( \Gamma \) denotes the unit circle and the function density \( \varphi \) is given by the following expression
\[ \varphi(t) = \frac{-2t^2 + 8t + 12}{4t(t^2 - t - 6)}. \]

\[
\begin{array}{|c|c|c|c|}
\hline
N & \| I - \bar{I} \|_1 & \| I - \bar{I} \|_2 & \| I - \bar{I} \|_\infty \\
\hline
20 & 1.8246599E-02 & 9.1665657E-03 & 5.0822943E-03 \\
40 & 3.6852972E-03 & 1.9270432E-03 & 1.5697196E-03 \\
60 & 2.3687426E-03 & 1.1880117E-03 & 6.6070486E-04 \\
\hline
\end{array}
\]
References


