Endpoint Values of Wavelets on an Interval

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Abstract

The aim of this work is to find an approximative method for computing the scaling functions constructed at the endpoints of an interval, using the inner product in \( L^2([0, 1]) \) of the scaling function and its derivative.

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Secondary 15A18.

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Introduction

As it is known, on the interval \([0, 1]\) the values of the scaling functions, the wavelets at the endpoints are not zero [2]. Unfortunately, there exists no method to calculate them due to the homogeneous algebraic system obtained by the relation (4) [5].

The idea of this work is to make a little detour, by making use of the scalar product in \( L^2([0, 1]) \) of the scaling function constructed on an interval and its derivative, to solve the algebraic system obtained in order to have a better approximation of these functions at the endpoints 0 and 1.

Multiresolution analysis

A multiresolution analysis on \( L^2([0, 1]) \) is given by an increasing sequence \( \{V_j\}_{j \geq j_0} \), \( j, j_0 \in \mathbb{Z} \), of closed subspaces of \( L^2([0, 1]) \) satisfying the following properties:

\[
\bigcup_{j=j_0}^{\infty} V_j \text{ is dense in } L^2([0, 1]),
\]

(1)
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\[ \forall f \in L^2([0, 1]), \forall j, j_0 \in \mathbb{Z}, j \geq j_0, \text{ we have } f(x) \in V_j \Rightarrow f(2x) \in V_{j+1}, \quad (2) \]

\[ \{ \Phi_{jk}, j \geq j_0, k = 0, 1, \ldots, 2^j - 1 \} = \{ \phi_{jk}^L, k = 0, 1, \ldots, N - 1 \} \quad (3) \]

\[ \cup \{ \phi_{jk}, k = N, \ldots, 2^j - N - 1 \} \cup \{ \phi_{jk}^R, k = -N, \ldots, -1 \} \]

is a system representing an orthonormal basis of \( V_j \), where

\[ \phi_{jk}^L(x) = 2^j \phi_k^L(2^j x), \quad \phi_{jk}(x) = 2^j \phi(2^j x - k), \quad \phi_{jk}^R(x) = 2^j \phi_k^R(2^j x) \]

are, respectively, the scaling functions of the endpoint 0, the internal ones and of the endpoint 1, given by,

for \( x \geq 0, k = 0, 1, \ldots, N - 1 \), we have \( \Phi_k(x) = \phi_k^L(x) \),

\[ \phi_k^L(x) = \sqrt{2} \sum_{l=0}^{N-1} H_{k,l}^L \phi_l^L(2x) + \sqrt{2} \sum_{m=N}^{N+2k} h_{k,m} \phi(2x - m); \quad (4) \]

for \( x \geq 0, k = N, \ldots, 2^j - N - 1 \), we have \( \Phi_k(x) = \phi_k(x) \),

\[ \phi_k(x) = \phi(x - k) = \sqrt{2} \sum_{q=-N+1}^{N} h_q \phi(2x - 2k - q); \quad (5) \]

for \( x \leq 0, k = -N, \ldots, -1 \), we have \( \Phi(x) = \phi_k^R(x) \),

\[ \phi_k^R(x) = \sqrt{2} \sum_{l=-N}^{-1} H_{k,l}^R \phi_l^R(2x) + \sqrt{2} \sum_{m=2k-N+1}^{N-1} h_{k,m} \phi(2x - m). \quad (6) \]

In an analogous way, we define \( W_j \), the complementary subspace of \( V_j \) in \( V_{j+1} \), as the subspace generated by the orthonormal basis

\[ \{ \Psi_{jk}, j \geq j_0, k = 0, 1, \ldots, 2^j - 1 \} = \{ \psi_{jk}^L, k = 0, 1, \ldots, N - 1 \} \]

\[ \cup \{ \psi_{jk}, k = N, \ldots, 2^j - N - 1 \} \cup \{ \psi_{jk}^R, k = -N, \ldots, -1 \}, \]

where

\[ \psi_{jk}^L(x) = 2^j \psi_k^L(2^j x), \quad \psi_{jk}(x) = 2^j \psi(2^j x - k), \quad \psi_{jk}^R(x) = 2^j \psi_k^R(2^j x) \]

are, respectively, the wavelets of the endpoint 0, the internal ones and of the endpoint 1, satisfying the algebraic systems,

for \( x \geq 0, k = 0, 1, \ldots, N - 1 \), we have \( \Psi_k(x) = \psi_k^L(x) \),

\[ \psi_k^L(x) = \sqrt{2} \sum_{l=0}^{N-1} G_{k,l}^L \phi_l^L(2x) + \sqrt{2} \sum_{m=N}^{N+2k} g_{k,m} \phi(2x - m); \quad (4') \]
for \( x \geq 0, k = N, \ldots, 2^j - N - 1 \), we have \( \Psi_k(x) = \psi_k(x) \),
\[
\psi_k(x) = \psi(x - k) = \sqrt{2} \sum_{q=-N+1}^{N} g_q \phi(2x - 2k - q); \tag{5'}
\]
pour \( x \leq 0, k = -N, \ldots, -1 \), we have \( \Psi(x) = \psi^R_k(x) \),
\[
\psi^R_k(x) = \sqrt{2} \sum_{l=-N}^{-1} G^R_{k,l} \phi^R_l(2x) + \sqrt{2} \sum_{m=2k-N+1}^{-N-1} g^R_{k,m} \phi(2x - m). \tag{6'}
\]

From the relation \( V_{j+1} = V_j \oplus W_j \), and from (1), we obtain
\[
V_J \oplus \bigoplus_{j=J}^{\infty} W_j = L^2([0, 1]), \quad J \geq J_0, \quad 2^{J_0} \geq 2N,
\]
which gives the expansion of any function \( f \) of \( L^2([0, 1]) \),
\[
f(x) = P_J f(x) + \sum_{j=J}^{\infty} Q_j f(x), \tag{7}
\]
where \( P_J \) is the orthogonal projection of the function \( f \) onto \( V_J \) and \( Q_j \) its orthogonal projection onto \( W_j \).

**Moments of the scaling functions on an interval**

Denote by \( m^L_{k,i} \) the moment of order \( i \) of the function \( \phi^L_k \) defined by
\[
m^L_{k,i} = \int_0^{\infty} x^i \phi^L_k(x) \, dx.
\]
From the scaling relation (4), we obtain for \( i = 0 \) the following algebraic system
\[
m^L_{k,0} = \sqrt{2} \sum_{i=0}^{N-1} H^L_{k,i} m^L_{k,0} + \sqrt{2} \sum_{m=N}^{N+2k} h^L_{k,m} \frac{M_0}{2},
\]
where \( M_0 = 1 \). In fact, we have \( m \geq N \), hence
\[
\int_0^{\infty} \phi(x - m) \, dx = \int_{-m}^{\infty} \phi(x) \, dx = \int_{-\infty}^{+\infty} \phi(x) \, dx = 1,
\]
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which implies that for different values of \( i \), we have the following algebraic system [8],

\[
2^i \sqrt{2} m_{k,i}^L = \sum_{l=0}^{N-1} H_{k,l,m}^L m_{k,i}^L + \sum_{m=N}^{N+2k} h_{k,m}^L \left( \sum_{j=0}^i \binom{i}{j} m^j M_{i-j} \right),
\]

where \( M_p \) is the moment of order \( p \) of the scaling function on \( \mathbb{R} \) and \( \binom{i}{j} = \frac{i!}{j!(i-j)!} \).

In the same way, we find the moments \( m_{k,i}^R \) of the scaling functions \( \phi_k^R \) [8],

\[
2^i \sqrt{2} m_{k,i}^R = \sum_{l=-N}^{-1} H_{k,l,m}^R m_{k,i}^R + \sum_{m=2k-N+1}^{-N-1} h_{k,m}^R \left( \sum_{j=0}^i \binom{i}{j} m^j M_{i-j} \right).
\]

**Applications**

The moments of the scaling functions \( \phi^L, \phi^R \) et \( \phi \) for Daubechies’s wavelets \( D_4 \) with two vanishing moments [8, 9], with \( i = 0, 1, 2 \), are

<table>
<thead>
<tr>
<th>( i )</th>
<th>( m_{0,i}^L )</th>
<th>( m_{1,i}^L )</th>
<th>( m_{-2,i}^R )</th>
<th>( m_{-1,i}^R )</th>
<th>( M_i )</th>
</tr>
</thead>
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<tr>
<td>0</td>
<td>0.3620521</td>
<td>1.001445</td>
<td>1.089843</td>
<td>1.295480</td>
<td>1.000000</td>
</tr>
<tr>
<td>1</td>
<td>-0.1509356</td>
<td>1.032428</td>
<td>-1.995769</td>
<td>-0.7217156</td>
<td>0.6339746</td>
</tr>
<tr>
<td>2</td>
<td>-0.3873851</td>
<td>1.166270</td>
<td>3.499148</td>
<td>0.5874012</td>
<td>0.4019238</td>
</tr>
</tbody>
</table>

The moments of the scaling functions on an interval for \( N = 2 \) (\( D_4 \) Daubechies)

**Scalar product of the scaling functions on an interval**

Denote by \( \theta_{jkl} \) the matrix of the scalar product \( \langle \Phi_j^l, \Phi_{j'}^l \rangle \) given by

\[
\theta_{jkl} = \int_0^1 \Phi_j^l(x) \Phi_{j'}^l(x) \, dx = 2^i \int_0^{2^j} \Phi_k(x) \Phi_{l}(x) \, dx = 2^i \theta_{kl},
\]

let \( \Theta \) be the matrix defined by \( \Theta = \theta_{kl} \), where

\[
\theta_{kl} = \langle \Phi_k^l, \Phi_l \rangle = \int_0^\infty \Phi_k^l(x) \Phi_l(x) \, dx
\]
The construction of the scaling functions on an interval shows us that the matrix \( \Theta = \theta_{kl} \) depends on nine submatrices given as follows,

\[
\Theta = \theta_{kl} = \begin{pmatrix}
\theta_{LL} & \theta_{LI} & \theta_{LR} \\
\theta_{IL} & \theta_{II} & \theta_{IR} \\
\theta_{RL} & \theta_{RI} & \theta_{RR}
\end{pmatrix},
\]

\( k = 0, 1, \ldots, N - 1, \)
\( l = 0, 1, \ldots, N - 1, \)

\[
\theta_{LL} = \int_{0}^{\infty} \phi_{0}^{L}(x) \phi_{0}^{L}(x) \, dx,
\]

\( k = 0, 1, \ldots, N - 1, \)
\( l = N, \ldots, 2^j - N - 1, \)

\[
\theta_{LI} = \int_{0}^{\infty} \phi_{k}^{L}(x) \phi(x - l) \, dx,
\]

\( k = N, \ldots, 2^j - N - 1, \)
\( l = 0, 1, \ldots, N - 1, \)

\[
\theta_{II} = \int_{0}^{\infty} \phi'(x - k) \phi_{l}^{L}(x) \, dx,
\]

\( k = N, \ldots, 2^j - N - 1, \)
\( l = N, \ldots, 2^j - N - 1, \)

\[
\theta_{RI} = \int_{0}^{\infty} \phi'(x - k + 2^j) \phi_{l}^{R}(x) \, dx,
\]

\( k = -N, \ldots, -1, \)
\( l = N, \ldots, 2^j - N - 1, \)

\[
\theta_{RR} = \int_{0}^{\infty} \phi_{k}^{R}(x) \phi(x - l + 2^j) \, dx,
\]

\( k = -N, \ldots, -1, \)
\( l = -N, \ldots, -1, \)

\[
\theta_{RL} = \int_{-\infty}^{0} \phi_{k}^{R}(x) \phi_{l}^{R}(x) \, dx.
\]

The above relation between the scaling functions shows that the elements of the submatrices \( \theta_{LL}, \theta_{LI}, \ldots, \theta_{RR} \) depend on the elements of the internal block \( \theta_{II} \) of the matrix.
Lemma 1 The matrix $\Theta = \theta_{kl}$ is a band matrix with a half band width $2N - 1$.

In fact, due to the compact and disjoint supports of the functions of the endpoints $\phi_{k}^{L}$, $\phi_{k}^{R}$, we have

$$\theta_{LR}^{IL} = \int_{0}^{\infty} \phi_{k}^{IL}(x)\phi_{k}^{IR}(x) \, dx = 0,$$

as well as

$$\theta_{RL}^{IL} = \int_{-\infty}^{0} \phi_{k}^{IR}(x)\phi_{k}^{IL}(x) \, dx = 0.$$

Lemma 2 The elements of the internal matrix $\theta^{II}$ are antisymmetric,

$$\theta_{kl}^{II} = -\theta_{lk}^{II},$$

moreover, they satisfy the algebraic system

$$\theta_{kl}^{II} = \theta_{2k,2l}^{II} + \frac{1}{2} \sum_{r=1}^{N} d_{2r-1}(\theta_{2k,2l+2r-1}^{II} + \theta_{2k+2r-1,2l}^{II}).$$  \hspace{1cm} (8)

In fact, applying to the expression

$$\theta_{kl}^{II} = \int_{0}^{\infty} \phi'(x-k)\phi(x-l) \, dx$$  \hspace{1cm} (9)

an integration by parts, we obtain

$$\theta_{kl}^{II} = \phi(x-k)\phi(x-l) \big|_{0}^{\infty} - \int_{0}^{\infty} \phi'(x-l)\phi(x-k) \, dx = -\theta_{lk}^{II},$$

moreover, the scaling relation (5) applied to equation (9) leads us directly to system (8); found also in [1].

Lemma 3 The elements of the matrix $\theta^{LL}$ satisfy the algebraic system

$$\theta_{kl}^{LL} = 2 \sum_{p=0}^{N-1} \sum_{q=-N+1}^{N} H_{kp}^{L} h_{q}^{L} \theta_{p,2l+q}^{LL} + 2 \sum_{m=N}^{N+2k} \sum_{q=-N+1}^{N} h_{km}^{L} h_{q}^{L} \theta_{m,2l+q}^{LL}.$$  \hspace{1cm} (10)

Moreover, we have

$$\theta_{kl}^{LL} = -\theta_{lk}^{IL}.$$
In fact, let
\[ \theta_{kl}^{LI} = \int_0^\infty \phi_k^L(x) \phi_l(x - l) \, dx. \quad (11) \]
The scaling relations (4) and (5) applied to equation (11) give the system (10); an integration by parts of this equation leads us to the relation \( \theta_{kl}^{LI} = -\theta_{lk}^{IL} \).

Let us note that the calculation of the elements de la matrix \( \theta^{LI} \) is done in a simple and straightforward way, beginning with the elements \( \theta_{kl}^{LI} \) such that \( 2l + q > 3N - 3 \), for all \( k = 0, 1, \ldots, N - 1 \), and in an analogous way we find the matrices \( \theta^{RI} \) and \( \theta^{IR} \).

**Lemma 4** The elements of the matrix \( \theta^{LL} \) satisfy the algebraic system
\[
\begin{align*}
\theta_{kl}^{LL} &= 2 \sum_{p=0}^{N-1} \sum_{q=0}^{N-1} H_{kp}^L H_{lq}^L \phi_{pq}^{LL} + 2 \sum_{p=0}^{N-1} \sum_{s=N}^{N+2l} H_{kp}^L h_{ls}^L \theta_{ps}^{II} \\
&+ 2 \sum_{m=N}^{N+2k} \sum_{q=0}^{N-1} h_{km}^L H_{lq}^L \theta_{mq}^{II} + 2 \sum_{m=N}^{N+2k} \sum_{s=N}^{N+2l} h_{km}^L h_{ls}^L \theta_{ms}^{II},
\end{align*}
\]
moreover, we have the relation
\[ \theta_{kl}^{LL} + \theta_{lk}^{LL} = -\phi_k^L(0) \phi_l^L(0). \]

In fact, let
\[ \theta_{kl}^{LL} = \int_0^\infty \phi_k^L(x) \phi_l^L(x) \, dx. \quad (13) \]
The scaling relation (4) applied to equation (13) gives the algebraic system (12) formed by \( N \times N \) equations. The second term in (13) depends on elements of the matrices \( \theta^{LI}, \theta^{IL} \) and \( \theta^{II} \); found also in [5].

If we apply an integration by parts to the equation (13), we obtain
\[ \theta_{kl}^{LL} = \phi_k^L(x) \phi_l^L(x) \big|_0^\infty - \int_0^\infty \phi_l^L(x) \phi_k^L(x) \, dx, \]
thus
\[ \theta_{kl}^{LL} = -\phi_k^L(0) \phi_l^L(0) - \theta_{lk}^{IL}. \]

**Corollary 1** The values of the scaling functions at the endpoint of an interval are given by
\[ \phi_k^L(0) = \pm \sqrt{-2\theta_{kk}^{LL}}. \quad (14) \]
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It suffices to take \( k = l \) in the above expression.
In the same way we find

\[
\phi_k^R(0) = \pm \sqrt{2 \theta_{kk}^{RR}}. \tag{14'}
\]

We should notice that the calculation of the function \( \phi_k^L \) at the point 0 using the scaling relation (4) is practically impossible, due to the homogeneity of the system of equations; on the contrary, the relation (14) gives the approximate values of the scaling function at the endpoint, of course, after having solved the algebraic system (12). The same reasoning applies to the function \( \phi_k^R \) at the point 1.

Let

\[
\theta_{LL} = \begin{pmatrix}
\theta_{00}^{LL} & \theta_{01}^{LL} & \cdots & \theta_{0,N-1}^{LL} \\
\theta_{10}^{LL} & \theta_{11}^{LL} & \cdots & \theta_{1,N-1}^{LL} \\
\vdots & \vdots & \ddots & \vdots \\
\theta_{N-1,0}^{LL} & \theta_{N-1,1}^{LL} & \cdots & \theta_{N-1,N-1}^{LL}
\end{pmatrix},
\]

also, one has

\[
\theta_{RR} = \begin{pmatrix}
\theta_{2j-N,2j-N}^{RR} & \theta_{2j-N,2j-N+1}^{RR} & \cdots & \theta_{2j-N,2j-1}^{RR} \\
\theta_{2j-N+1,2j-N}^{RR} & \theta_{2j-N+1,2j-N+1}^{RR} & \cdots & \theta_{2j-N+1,2j-1}^{RR} \\
\vdots & \vdots & \ddots & \vdots \\
\theta_{2j-N+1,2j-N}^{RR} & \theta_{2j-N+1,2j-N+1}^{RR} & \cdots & \theta_{2j-N+1,2j-1}^{RR}
\end{pmatrix}.
\]

Applications

\[
\theta_2^{LL} = \begin{pmatrix}
-1.96344 & -1.52529 \\
0.93546 & -0.04429
\end{pmatrix}, \quad \theta_2^{RR} = \begin{pmatrix}
0.08996 & -0.79412 \\
0.31493 & 0.63712
\end{pmatrix}.
\]

It is easy to see that the scaling functions \( \phi_0^L \) and \( \phi_1^L \) are positives in a neighbourhood of 0, while the functions \( \phi_0^R \) and \( \phi_1^R \) are of the opposite sign, hence from the relations (14) and (14') we have

\[
\phi_0^L(0) = 1.981636, \quad \phi_1^L(0) = 0.2976239, \\
\phi_0^R(0) = -0.4241698, \quad \phi_1^R(0) = 1.128822.
\]

For the different values of \( N \) (the number of vanishing moments of the wavelets on an interval), one proceeds in the same way to obtain the approximate values of the scaling functions at the endpoints. The values of the wavelets at the endpoints are deduced in a very simple way.
References


