Nonexistence of Solutions for Semilinear Equations and Systems in Cylindrical Domains

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Abstract

We establish an integral identity in \( \Omega = \mathbb{R} \times ]0,1[ \) which we use to prove nonexistence of nontrivial solutions in \( H^2(\Omega) \cap L^\infty(\Omega) \) to some semilinear equations under some conditions on \( f \) and \( g \). We then extend this method to systems of the form

\[
\begin{align*}
\lambda \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} &= g(v) \quad \text{in} \quad \Omega = \mathbb{R} \times \mathbb{R}^+,
\lambda \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} &= f(u) \quad \text{in} \quad \Omega = \mathbb{R} \times \mathbb{R}^+,
u = v = 0 \quad \text{on} \quad \partial\Omega.
\end{align*}
\]

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1 Introduction and notations

The question of existence and the nonexistence of solutions for the semi-linear elliptic problem in bounded or unbounded domain \( \Omega \) in \( \mathbb{R}^N \)

\[
\left\{ \begin{array}{l}
-\Delta u + f(u) = 0 \quad \text{in} \quad \Omega, \\
u = 0 \quad \text{on} \quad \partial\Omega,
\end{array} \right.
\]

was studied by several authors for different reasons. We quote by way of examples the works of Esteban & Lions [2], Kirane, Nabana & Pohozaev [5], Pucci & Serrin [11], Pohozaev [12] and Van der Vorst [13].

M. J. Esteban & P.-L. Lions show that the Dirichlet problem

\[
\left\{ \begin{array}{l}
-\Delta u + f(u) = 0, \quad u \in C^2(\overline{\Omega}), \\
u = 0 \quad \text{on} \quad \partial\Omega,
\end{array} \right.
\]

472
satisfying $\nabla u \in L^2(\Omega)$, $F(u) = \int_0^u f(s)ds \in L^1(\Omega)$, where $\Omega$ is a connected unbounded domain of $\mathbb{R}^N$ such that

$$\exists \Lambda \in \mathbb{R}^N, |\Lambda| = 1, \langle n(x), \Lambda \rangle \geq 0 \text{ on } \partial \Omega, \langle n(x), \Lambda \rangle \neq 0$$

$(n(x)$ is the outward normal to $\partial \Omega$ at the point $x$) does not have a solution.

The question which arises then is to know if this result is still valid for the Neumann problem

$$\begin{cases} -\Delta u + f(u) = 0, \\
\frac{\partial u}{\partial n} = 0 \text{ on } \partial \Omega.
\end{cases}$$

The answer to this question is negative. Indeed, Berestycki, Gallouët and Kavian established that the problem

$$-\Delta u - u^3 + u = 0, \quad u \in H^2(\mathbb{R}^2),$$

admits a radial solution, see [1].

The same solution satisfies

$$\begin{cases} -\Delta u - u^3 + u = 0, \quad u \in H^2([0, +\infty[\times\mathbb{R}), \\
\frac{\partial u}{\partial n} = 0 \text{ on } \{0\} \times \mathbb{R}.
\end{cases}$$

To show the nonexistence of solutions of elliptic problems several methods exist, but for this work, we use integral identities.

We establish in the second section an integral identity in a cylindrical domain of $\mathbb{R}^2$ which shows that some semilinear elliptic as well as hyperbolic equations do not have nontrivial solutions in $H^2(\Omega) \cap L^\infty(\Omega)$.

In the third section, we illustrate our results by examples, namely we show that, under some assumptions on the nonlinearity, the Klein–Gordon equation does not have nontrivial solutions.

Finally, in the last section, we prove that with the help of two integral identities the following differential system

$$\begin{cases}
\lambda \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = g(v) \text{ in } \Omega = \mathbb{R} \times \mathbb{R}^+, \\
\lambda \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = f(u) \text{ in } \Omega = \mathbb{R} \times \mathbb{R}^+, \\
u = v = 0 \text{ on } \partial \Omega,
\end{cases}$$
where \( f \) and \( g \) satisfy
\[
\begin{cases}
 f, g \in C(\mathbb{R}), \\
 f(0) = g(0) = 0, \\
 F(u) \cdot G(v) \geq 0,
\end{cases}
\]
does not possess nontrivial solutions \((u, v)\) in \( H^2(\Omega) \cap L^\infty(\Omega) \times H^2(\Omega) \cap L^\infty(\Omega) \).

A nonexistence result for problems of the form
\[
\begin{cases}
 \Delta^2 u = f(u) \text{ in } \Omega, \\
 \Delta u = 0 \text{ on } \partial\Omega, \\
 u = 0 \text{ on } \partial\Omega,
\end{cases}
\]
will follow as a particular case of the above system.

Let us denote by \((x, y)\) a generic point of \( \Omega = \mathbb{R} \times ]\alpha, \beta[ \), \( \Gamma = \partial\Omega = \partial (\mathbb{R} \times ]\alpha, \beta[) \) \( = \mathbb{R} \times (\alpha) \cup \mathbb{R} \times (\beta) \) and \( n(x, y) = (n_1(x, y), n_2(x, y)) \) the outward normal to \( \Gamma \) at the point \((x, y)\). We consider a locally Lipschitzian real function
\[ f : ]\alpha, \beta[ \times \mathbb{R} \to \mathbb{R}, \]
such that \( f(y, 0) = 0 \ \forall y \in ]\alpha, \beta[ \), so that \( u = 0 \) is a solution of the problem
\[
\begin{cases}
 \lambda \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} + f(y, u) = 0 \text{ in } \Omega = \mathbb{R} \times ]\alpha, \beta[, \\
 u + \varepsilon \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega,
\end{cases}
\](P.1)
where \( \lambda \) is a real parameter and \( \varepsilon \) is a positive real number.

We shall also use the notation \( F(y, u) = \int_{\alpha}^{\beta} f(y,\sigma) d\sigma \).

## 2 General results

We are now in a position to state the following result:

**Proposition 1** Let \( u \) be a solution of (P.1), then for any \( x \in \mathbb{R} \) and \( \varepsilon > 0 \),
\[
\int_{\alpha}^{\beta} \left[ \lambda \left( \frac{\partial u}{\partial x} \right)^2 + \left( \frac{\partial u}{\partial y} \right)^2 + 2F(y, u) \right] (x, y) dy + \frac{1}{\varepsilon} \left[ u(x, \alpha)^2 + u(x, \beta)^2 \right] = 0. \tag{2.1}\]
Proof. Let us set
\[ K(x) = \int_{\alpha}^{\beta} \left[ \frac{\lambda}{2} \left| \frac{\partial u}{\partial x} \right|^2 + \frac{1}{2} \left| \frac{\partial u}{\partial y} \right|^2 + F(y, u) \right] (x, y) dy. \]

Under the above hypothesis \( K \) is absolutely continuous and we have almost everywhere on \( \mathbb{R} \):
\[ K_0(x) = \beta Z^\alpha \frac{\partial u}{\partial x} \frac{\partial^2 u}{\partial x^2} - \frac{d}{dy} \left( \frac{\partial u}{\partial y} \right) \frac{\partial u}{\partial x} + F(y, u) \] on \( \partial \Omega \). This yields
\[ K'(x) = \int_{\alpha}^{\beta} \left[ \lambda \left( \frac{\partial u}{\partial x} \right)^2 - \frac{d}{dy} \left( \frac{\partial u}{\partial y} \right) \frac{\partial u}{\partial x} + f(y, u) \right] (x, y) dy. \]

An integration by parts yields
\[ K'(x) = \left. \int_{\alpha}^{\beta} \left[ \lambda \left( \frac{\partial u}{\partial x} \right)^2 - \frac{d}{dy} \left( \frac{\partial u}{\partial y} \right) \frac{\partial u}{\partial x} + f(y, u) \right] (x, y) dy \right|_{y=\beta}^{y=\alpha} \]
\[ = \left. \left( \frac{\partial u}{\partial x} \right) \left( \frac{\partial u}{\partial y} \right) \right|_{y=\alpha}^{y=\beta}. \]

Or,
\[ u + \varepsilon \frac{\partial u}{\partial n} = 0 \text{ on } \partial \Omega \iff \begin{cases} \frac{\partial u(x, \beta)}{\partial y} + \frac{1}{\varepsilon} u(x, \beta) = 0, \\ \frac{\partial u(x, \alpha)}{\partial y} - \frac{1}{\varepsilon} u(x, \alpha) = 0. \end{cases} \]

If \( 0 < \varepsilon < +\infty \), we may write:
\[ \frac{\partial u(x, \beta)}{\partial y} = -\frac{1}{\varepsilon} u(x, \beta) \quad \text{and} \quad \frac{\partial u(x, \alpha)}{\partial y} = \frac{1}{\varepsilon} u(x, \alpha). \]

The boundary term is then equal to
\[ \left( \frac{\partial u}{\partial x} \right) \left( \frac{\partial u}{\partial y} \right) \bigg|_{y=\alpha}^{y=\beta} = -\frac{1}{\varepsilon} \left[ \left( \frac{\partial u(x, \beta)}{\partial x} \right) u(x, \beta) + \left( \frac{\partial u(x, \alpha)}{\partial x} \right) u(x, \alpha) \right] \]
\[ = -\frac{1}{2\varepsilon} \frac{d}{dx} \left[ (u(x, \alpha))^2 + (u(x, \beta))^2 \right], \]
and finally,
\[ \frac{d}{dx} \left( K(x) + \frac{1}{2\varepsilon} \left[ (u(x, \alpha))^2 + (u(x, \beta))^2 \right] \right) = 0, \]
thus the expression in parentheses is constant, but
\[ \int_{-\infty}^{+\infty} \left( K(x) + \frac{1}{2\varepsilon} \left[ (u(x, \alpha))^2 + (u(x, \beta))^2 \right] \right) \, dx < +\infty \]
implies that this constant is zero. This proves the Proposition. \[ \Box \]

**Remark 1** If \( \varepsilon = 0 \) (Dirichlet condition), \( u = 0 \) on \( \partial \Omega \) implies \( \nabla u = \frac{\partial u}{\partial n} \) and this allows us to write
\[ \left( \frac{\partial u}{\partial x} \right)(x, y) = \left( \frac{\partial u}{\partial n} \right) n_1(x, y) \]
and
\[ \left( \frac{\partial u}{\partial x} \right) \left( \frac{\partial u}{\partial y} \right) \bigg|_{y=\beta}^{y=\alpha} \]
vanishes.

If \( \varepsilon = +\infty \) (Neumann condition), \( \frac{\partial u}{\partial n} = 0 \) on \( \partial \Omega \) becomes
\[ \frac{\partial u}{\partial y} = 0 \quad \text{on} \quad \partial \Omega \]
and
\[ \left( \frac{\partial u}{\partial x} \right) \left( \frac{\partial u}{\partial y} \right) \bigg|_{y=\beta}^{y=\alpha} \]
also vanishes.

The problem (P.1) includes in fact **two types of equations** depending on whether \( \lambda \) is positive or negative.

### 2.1 Hyperbolic case

Let us present two theorems of nonexistence of nontrivial solutions.

**Theorem 1** Suppose that \( u \in H^2(\Omega) \cap L^\infty(\Omega) \) is a solution of (P.1), \( \lambda > 0 \) and \( f \) satisfies
\[ F(y, u) \geq 0. \] (A)
Then \( u \equiv 0 \).
Nonexistence of solutions for semilinear equations and systems

Proof. We apply formula (2.1) to obtain

$$
\int_{\alpha}^{\beta} \left( \frac{\lambda}{2} \left| \partial_u \frac{\partial}{\partial x} \right|^2 + \frac{1}{2} \left| \partial_u \frac{\partial}{\partial y} \right|^2 + F'(y,u) \right) (x,y) dy = 0.
$$

Let us multiply equation (P.1) by $\frac{1}{2} u$ and integrate over $]a, \beta[$ to obtain

$$
\int_{a}^{\beta} \left[ \frac{\lambda}{2} \left( \frac{\partial^2 u}{\partial x^2} \right) u - \frac{1}{2} \left( \frac{\partial^2 u}{\partial y^2} \right) u + \frac{1}{2} (f(y,u)) u \right] (x,y) dy = 0.
$$
which yields
\[
\int_{\alpha}^{\beta} \left[ \frac{\lambda}{2} \frac{\partial^2 (u^2)}{\partial x^2} - \frac{\lambda}{2} \left| \frac{\partial u}{\partial x} \right|^2 + 1 \left| \frac{\partial u}{\partial y} \right|^2 + \frac{1}{2} f(y, u)u \right] (x, y) dy = \frac{1}{2} \left. u \frac{\partial u}{\partial y} \right|_{y=\beta}^{y=\alpha} - \frac{1}{2\varepsilon} \left[ (u(x, \alpha))^2 + (u(x, \beta))^2 \right],
\]
which combined with (2.1) yields
\[
\frac{\lambda}{4} \frac{d^2}{dx^2} \left[ \int_{\alpha}^{\beta} |u(x, y)|^2 dy \right] = \int_{\alpha}^{\beta} \left[ \frac{\lambda}{2} \left| \frac{\partial u}{\partial x} \right|^2 + F(y, u) - \frac{1}{2} uf(y, u) \right] dy.
\]
The hypothesis (B) implies that
\[
\frac{\lambda}{4} \frac{d^2}{dx^2} \left[ \int_{\alpha}^{\beta} |u(x, y)|^2 dy \right] \geq \frac{\lambda}{4} \int_{\alpha}^{\beta} \left| \frac{\partial u}{\partial x} \right|^2 dy
\]
and \(\lambda > 0\) implies the desired result.

2.2 Elliptic equations

For the elliptic case, we have a nonexistence result stated in the following manner:

**Theorem 3** Let \(u \in H^2(\Omega) \cap L^\infty(\Omega)\) be a solution of (P.1), \(\lambda < 0\) and \(f\) satisfying
\[
2F(y, u) - uf(y, u) \leq 0, \quad y \in ]\alpha, \beta[.
\]
Then the function \(j(x)\) defined in Theorem 2 is convex on \(\mathbb{R}\).

**Proof.** Similar to the proof of Theorem 2.

3 Examples

In this section, we present some examples illustrating the preceding theorems.

**Example 1** Let \(\rho\) be a function of \(C^1, \rho \in ]\alpha, \beta[ \mapsto \mathbb{R}, \lambda \in \mathbb{R}\) and \(f(y, u) \equiv \rho(y)u\).

For \(u \in H^2(\Omega) \cap L^\infty(\Omega)\) , the problem
\[
\left\{
\begin{aligned}
\lambda \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} + \rho(y)u = 0 \quad &\text{in } \Omega, \\
u + \varepsilon \frac{\partial u}{\partial n} = 0 \quad &\text{on } \partial \Omega,
\end{aligned}
\right.
\]
does not have nontrivial solutions.
Example 2 Let us consider the **Klein-Gordon** equation

\[
\begin{cases}
\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} + mu - \theta_1 |u|^{p-1}u - \theta_2 |u|^{q-1}u = 0 \quad \text{in} \quad \Omega, \\
u + \varepsilon \frac{\partial u}{\partial n} = 0 \quad \text{on} \quad \partial \Omega
\end{cases}
\]

(3.2)

where \(m > 0\) is the mass of a particle, \(\theta_1, \theta_2\) are positive constants, \(p\) and \(q\) are numbers greater than one. The problem (3.2) does not possess nontrivial solutions in \(H^2(\Omega) \cap L^\infty(\Omega)\). It suffices to note that

\[
F(y, u) - \frac{1}{2}uf(y, u) = \theta_1 \left( \frac{1}{2} - \frac{1}{p+1} \right) |u|^{p+1} + \theta_2 \left( \frac{1}{2} - \frac{1}{q+1} \right) |u|^{q+1}.
\]

Example 3 Let \(\rho\) be a nonnegative function of class \(C^1\), and \(\omega\) a parameter, then the problem

\[
\begin{cases}
-\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} + \rho(y) \left( \omega u + |u|^{\tau-1}u \right) = 0 \quad \text{in} \quad \Omega, \\
u + \varepsilon \frac{\partial u}{\partial n} = 0 \quad \text{on} \quad \partial \Omega
\end{cases}
\]

(3.3)

does not possess nontrivial solutions in \(H^2(\Omega) \cap L^\infty(\Omega)\).

Remark 3 If \(\Omega = \mathbb{R} \times [\alpha, +\infty[\), \(\alpha \in \mathbb{R}\), we may get results on nonexistence of solutions for the problem

\[
\begin{cases}
\lambda \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} + f(x, u) = 0 \quad \text{in} \quad \Omega, \\
u + \varepsilon \frac{\partial u}{\partial n} = 0 \quad \text{on} \quad \partial \Omega
\end{cases}
\]

\((P.1)')\)

We find that

\[
\int_{-\infty}^{+\infty} \left[ \frac{\lambda}{2} \left| \frac{\partial u}{\partial x} \right|^2 - \frac{1}{2} \left| \frac{\partial u}{\partial y} \right|^2 + F(x, u) \right] (x, y) \, dx = 0.
\]

Probably it would be interesting to study the problem

\[
\begin{cases}
\lambda \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} + f(x, y, u) = 0 \quad \text{in} \quad \Omega, \\
u + \varepsilon \frac{\partial u}{\partial n} = 0 \quad \text{on} \quad \partial \Omega.
\end{cases}
\]

\((P)\)
4 Application to differential systems

In this last section we study both elliptic and hyperbolic differential systems in \( \Omega = \mathbb{R} \times \mathbb{R}^+ \). Pucci & Serrin [11] and Van der Vorst [13] have studied elliptic systems on star-shaped domains in \( \mathbb{R}^N \). Van der Vorst showed that

\[
\begin{cases}
\Delta u = g(v) \text{ in } \Omega, \\
\Delta v = f(u) \text{ in } \Omega, \\
u = v = 0 \text{ on } \partial \Omega,
\end{cases}
\]

where \( f \) and \( g \) satisfy

\[
\begin{cases}
f, g \in C(\mathbb{R}), \\
f(u) > 0 \text{ if } u > 0; \ f(u) < 0 \text{ if } u < 0; \ f(0) = 0; \ NF(u) - a_1u f(u) \leq 0, \ u \neq 0, \\
g(v) > 0 \text{ if } v > 0; \ g(v) < 0 \text{ if } v < 0; \ g(0) = 0; \ NG(v) - a_2v g(v) \leq 0, \ v \neq 0, \\
N - a_1 - a_2 \leq 0,
\end{cases}
\]

do not possess nontrivial solutions in \( C^2(\Omega) \cap C^1(\overline{\Omega}) \).

We consider the system

\[
\begin{cases}
\lambda \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = g(v) \text{ in } \Omega = \mathbb{R} \times \mathbb{R}^+, \\
\lambda \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = f(u) \text{ in } \Omega = \mathbb{R} \times \mathbb{R}^+, \\
u = v = 0 \text{ on } \partial \Omega,
\end{cases}
\]

(P.2)

where \( f \) and \( g \) satisfy the following hypothesis:

\[
\begin{cases}
f, g \in C(\mathbb{R}), \\
f(0) = g(0) = 0.
\end{cases}
\]

We have

**Proposition 2** Let \( \lambda \in \mathbb{R} \) and \( (u, v) \in H^2(\Omega) \cap L^\infty(\Omega) \times H^2(\Omega) \cap L^\infty(\Omega) \) be a solution of problem (P.2), then, almost everywhere on \( \mathbb{R} \),

\[
\int_{0}^{+\infty} \left[ \left( \frac{\partial u}{\partial y} \right) \left( \frac{\partial v}{\partial y} \right) - \lambda \left( \frac{\partial u}{\partial x} \right) \left( \frac{\partial v}{\partial x} \right) + G(v) + F(u) \right] (x, y) \, dy = 0, \quad (4.1)
\]

and almost everywhere on \( \mathbb{R}^+ \)

\[
\int_{-\infty}^{+\infty} \left[ - \left( \frac{\partial u}{\partial y} \right) \left( \frac{\partial v}{\partial y} \right) + \lambda \left( \frac{\partial u}{\partial x} \right) \left( \frac{\partial v}{\partial x} \right) + G(v) + F(u) \right] (x, y) \, dx = 0. \quad (4.2)
\]
**Theorem 4** Assume that \( f \) and \( g \) satisfy

\[
F(u) \cdot G(v) \geq 0. \tag{C}
\]

Then the problem \((P.2)\) does not possess nontrivial solutions \((u, v)\) in \( H^2(\Omega) \cap L^\infty(\Omega) \times H^2(\Omega) \cap L^\infty(\Omega) \).

**Proof of Theorem 4.** From formulae \((4.1)\) and \((4.2)\), we obtain

\[
\int_\Omega \left[ \left( \frac{\partial u}{\partial y} \right) \left( \frac{\partial v}{\partial y} \right) - \lambda \left( \frac{\partial u}{\partial x} \right) \left( \frac{\partial v}{\partial x} \right) + G(v) + F(u) \right] (x, y) \, dx \, dy = 0
\]

and

\[
\int_\Omega \left[ - \left( \frac{\partial u}{\partial y} \right) \left( \frac{\partial v}{\partial y} \right) + \lambda \left( \frac{\partial u}{\partial x} \right) \left( \frac{\partial v}{\partial x} \right) + G(v) + F(u) \right] (x, y) \, dx \, dy = 0.
\]

Adding both formulae, we find

\[
\int_\Omega [G(v) + F(u)] (x, y) \, dx \, dy = 0.
\]
Hypothesis (C) implies that
\[ F(u) = 0 \quad \text{in} \quad \Omega \]
and
\[ G(v) = 0 \quad \text{in} \quad \Omega. \]

As in [6, Theorem 1], the problem \((P.2)\) becomes
\[
\begin{align*}
\lambda \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} &= 0 \quad \text{in} \quad \Omega = \mathbb{R} \times \mathbb{R}^+, \\
\lambda \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} &= 0 \quad \text{in} \quad \Omega = \mathbb{R} \times \mathbb{R}^+, \\
u = v = 0 &\quad \text{on} \quad \partial \Omega.
\end{align*}
\]

For any one of these equations we check that
\[
\int_0^{+\infty} \left[ \lambda \frac{\partial u}{\partial x} \frac{\partial u}{\partial y} - |\partial u|^2 \right] (x, y) \, dy = 0. \tag{4.3}
\]
The multiplication by \(u\) and integration over \([0, +\infty[\) yield
\[
\int_0^{+\infty} \left[ \frac{\lambda}{2} \frac{d^2}{dx^2} \left[ |u|^2 \right] - \lambda \left| \frac{\partial u}{\partial x} \right|^2 - \left| \frac{\partial u}{\partial y} \right|^2 \right] (x, y) \, dy = 0. \tag{4.4}
\]
Combining formulae (4.3) and (4.4), we get
\[
\lambda \frac{d^2}{dx^2} \left[ \int_0^{+\infty} |u(x, y)|^2 \, dy \right] = 4 \int_0^{+\infty} \left| \frac{\partial u}{\partial y} \right|^2 (x, y) \, dy \geq 0.
\]

If \(\lambda > 0\), we conclude as in Theorem 3.
If \(\lambda < 0\), (4.3) yields
\[
\frac{\partial u}{\partial x} (x, y) = 0 = \frac{\partial u}{\partial y} (x, y),
\]
and we conclude as in Theorem 2. \(\Box\)

**Example 4** Let \(g(v) = v\) and \(f(u)\) be such that such that \(F(u) \geq 0\), then the following problem
\[
\begin{align*}
\Delta^2 u = f(u) &\quad \text{in} \quad \Omega = \mathbb{R} \times \mathbb{R}^+, \\
\Delta u = 0 &\quad \text{on} \quad \partial \Omega, \\
u = 0 &\quad \text{on} \quad \partial \Omega,
\end{align*}
\]
\((P.2)')\]
does not have nontrivial solutions in \(H^2(\Omega) \cap L^\infty(\Omega)\).
**Nonexistence of solutions for semilinear equations and systems**

**Proof.** Let

\[ \Delta u = v. \]

\((P.2)’\) reduces to

\[
\begin{cases}
\Delta u = v \text{ in } \Omega = \mathbb{R} \times \mathbb{R}^+,
\Delta v = f(u) \text{ in } \Omega = \mathbb{R} \times \mathbb{R}^+,
\end{cases}
\]

\[ u = \Delta u = 0 \text{ on } \partial \Omega. \]

The conclusion follows from Theorem 4. \(\square\)

**Example 5** Let

\[ f(u) = u(u + a)(u + b) \text{ with } \begin{aligned} ab &\geq \frac{2}{5}(a^2 + b^2), \\
&\quad a, b \in \mathbb{R} \end{aligned} \]

and

\[ g(v) = v. \]

The system \((P.2)\) does not possess nontrivial solutions, and it is clear that the result of Van der Vorst does not permit to conclude it.
References


