Contemporary Theory of Functional Differential Equations and Some Classical Problems

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A survey of the recent trends and ideas of the Perm Seminar

Let $D$ be a linear manifold of elements $x$, $B$ be a Banach space of elements $z$, $\mathbb{R}^n$ be the space of vectors $\alpha = \text{col}\{\alpha_1, \ldots, \alpha_n\}$ with real elements. Further assume that there exists a pair of linear operators $\delta : D \to B$ and $r : D \to \mathbb{R}^n$ such that the systems of equations

$$\delta x = z, \quad rx = \alpha,$$

has a unique solution $x \in D$ for any $z \in B$, $\alpha \in \mathbb{R}^n$. We will write down the solution of (1) in the form

$$x = Wz + U\alpha.$$

The “$W$–substitution” (2) puts an element $x \in D$ in correspondence to any pair $(z, \alpha) \in B \times \mathbb{R}^n$ and to any $x \in D$ it puts in correspondence a pair $z \in B$ and $\alpha \in \mathbb{R}^n$ by (1).

Thus the manifold $D$ is isomorphic to the direct product $B \times \mathbb{R}^n$ ($D \simeq B \times \mathbb{R}^n$). The linear isomorphism $J = \{W, U\} : B \times \mathbb{R}^n \to D$ is defined by the operators $W : B \to D$ and $U : \mathbb{R}^n \to D$. The space $D$ is a Banach one under the norm $\|x\|_D = \|\delta x\|_B + \|rx\|_{\mathbb{R}^n}$. An example of such a $D$ is the space of absolutely continuous functions (the functions $x : [a, b] \to \mathbb{R}^n$ such that $x(t) = \int_a^b \dot{x}(s) \, ds + x(a)$). Here

$$(Wz)(t) = \int_a^t z(s) \, ds, \quad U\alpha = \alpha,$$

$B$ is the space $L$ of summable $z : [a, b] \to \mathbb{R}^n$.

The equation

$$\dot{x} = Fx$$

in the space $D$ of absolutely continuous functions was thoroughly studied by the Perm Seminar in [1, 2]. This equation is called “the functional differential” one. It is a natural generalization of the classical ordinary differential equation and contains in itself the equations with deviating argument, integro-differential and many other with respect to a differentiable unknown $x$. 

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The theory of $\dot{x} = Fx$ is founded on the isomorphism $\mathcal{J} : \mathbb{L} \times \mathbb{R}^n \to \mathbb{D}$ and the specific character of the Lebesgue space $\mathbb{L}$ shows itself only in some questions. On the whole, the fact that $\mathbb{L}$ is a Banach space was used.

Therefore in replacement of the space $\mathbb{L}$ by an arbitrary Banach space $\mathbb{B}$ most of the fundamental statements of the general theory of equation (3) still hold. Thus a further generalization of the differential equation arises. Namely, the theory of the equation $Lx = Fx$ in the space $\mathbb{D} \simeq \mathbb{B} \times \mathbb{R}^n$ with a linear operator $\mathcal{L} : \mathbb{D} \to \mathbb{B}$ and a compact operator $F : \mathbb{D} \to \mathbb{B}$. To the theory of such a generalization the monographs [3, 4] are devoted.

In the mentioned monographs the role of the choice of the adequate space to the problem considered is emphasized. Moreover, there are proposed new effective approaches to some classical problems on the base of the general theory. Let us make myself more explicit using some examples.

1 Stability and asymptotic behavior of solutions

Let $\mathbb{D} \simeq \mathbb{B} \times \mathbb{R}^n$ be a space of functions with given asymptotics. We say that the linear equation $Lx = f$ with the linear operator $\mathcal{L} : \mathbb{D} \to \mathbb{B}$ is $\mathbb{D}$–stable if the Cauchy problem

$$Lx = f, \quad x(0) = \alpha$$

has a unique solution $x \in \mathbb{D}$ for each pair $\{f, \alpha\} \in \mathbb{B} \times \mathbb{R}^n$.

Thus, the equation $Lx = f$ (for instance $Lx \overset{\text{def}}{=} \dot{x} + Ax$) is exponentially stable, if problem (4) is correctly solvable in the space $\mathbb{D} \simeq \mathbb{L}^\infty \times \mathbb{R}^n$, with elements defined by

$$x(t) = \int_0^t e^{s-t} z(s) \, ds + \alpha e^{-t}, \quad z \in \mathbb{L}^\infty$$

(by the $W$–substitution (2) $x = Wz + U\alpha$ with $(Wz)(t) = \int_0^t e^{s-t} z(s) \, ds$, $(U\alpha)(t) = \alpha e^{-t}$).

The general theory offers some theorems on solvability of problem (4). For instance,

**Theorem.** Let $\mathcal{L} : \mathbb{D} \to \mathbb{B}$ be bounded. Problem (4) is correctly solvable if and only if the operator $\mathcal{L}W : \mathbb{B} \to \mathbb{B}$ has the bounded inverse $[\mathcal{L}W]^{-1}$ ([4]–[6]).

Let us emphasize that the operator $\mathcal{L}W$ has an explicit form and by means of this theorem we obtain both known and some new effective tests of stability.
2 A new effective approach to some problems of the calculus of variations

Let $\mathbf{L}_2$ be the Banach space of functions $z : [a, b] \to \mathbb{R}^1$ under the norm

$$\|z\|_{\mathbf{L}_2} = \left( \int_a^b z^2(s) \, ds \right)^{1/2}$$

and $\mathbf{D}$ be isomorphic to $\mathbf{L}_2 \times \mathbb{R}^n$. Consider the problem of minimization in the space $\mathbf{D}$ of the functional with linear restrictions

$$I(x) = b \int_a^b f(s, (T_1 x)(s), \ldots, (T_m x)(s)) \, ds \to \min,$$

$$\ell x \overset{\text{def}}{=} \text{col}\{\ell^1 x, \ldots, \ell^n x\} = \alpha \overset{\text{def}}{=} \text{col}\{\alpha^1, \ldots, \alpha^n\}.$$  \hspace{1cm} (5)

Here $T_i : \mathbf{D} \to \mathbf{L}_2$ and $\ell^i : \mathbf{D} \to \mathbb{R}^1$ are linear bounded operators.

Let us choose the linear operator $\delta : \mathbf{D} \to \mathbf{L}_2$ such that the problem

$$\delta x = z, \quad \ell^i x = \alpha^i, \quad i = 1, \ldots, n,$$  \hspace{1cm} (1 bis)

has a unique solution

$$x = Wz + U\alpha.$$  \hspace{1cm} (2 bis)

Let us further construct an auxiliary functional $I_1(z) \overset{\text{def}}{=} I(Wz + U\alpha)$ in the space $\mathbf{L}_2$. If $x_0 \in \mathbf{D}$ is a point of minimum to problem (5), then $z_0 = \delta x_0$ is a point of minimum to the functional $I_1(z)$. And if $z_0 \in \mathbf{L}_2$ is a point of minimum of the functional $I_1(z)$, then $x_0 = Wz_0 + U\alpha$ ($\alpha = \text{col}\{\alpha^1, \ldots, \alpha^n\}$) is a point of minimum of problem (5).

Thus the $W-$substitution (2 bis) reduces problem (5) in the space $\mathbf{D} \simeq \mathbf{L}_2 \times \mathbb{R}^n$ to the much simpler one in the “convenient” space $\mathbf{L}_2$. It is relevant to say that the space $\mathbf{L}_2$ is very convenient for problems of the calculus of variations. In particular, the theorems of Fermat and Weierstrass on the minimum are still valid for functionals in the space $\mathbf{L}_2$. On the base of these theorems there effective tests for convexity of functionals are established.

3 Regularization of singular equations

One and the same equation may be singular in one space and regular in another. For instance, the equation

$$(\mathcal{L} x)(t) \overset{\text{def}}{=} t(1-t)\ddot{x}(t) + \frac{q(t)}{t} \dot{x}(t) + \frac{p(t)}{1-t} x(t) = f(x), \quad t \in [0, 1],$$

is singular at $t = 0$ and $t = 1$.
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with summable $p(\cdot)$ and $q(\cdot)$ is singular in the traditional space $W^2$ of $x : [0, 1] \rightarrow \mathbb{R}^1$ with absolutely continuous $\dot{x}$ and summable $\ddot{x}$. Nevertheless, it becomes regular (complies with a theorem of the general theory) in the space of $D \simeq L^\gamma \times \mathbb{R}^2$, where $L^\gamma$ is the weighted space $L^\gamma (z \in L^\gamma$ if the product $t(1-t)z(t)$ is summable).

The ideas and applications of the general theory had explicit influence on many other actual problems. In the works of G. G. Islamov [7, 8] an original view on the construction of the spaces $D \simeq B \times \mathbb{R}^n$ and on the problem of equations of the first order was proposed. The general theory had a very essential impact on our assumption about stochastic equations [1, 4].

Constructive methods, control theory and economic dynamics obtain many new approaches under the influence of the ideas and methods of the contemporary theory of functional differential equations [1, 4].

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References


