Asymptotic Behavior of the Solution, when $t \to +\infty$, of a Class of Nonlinear Equations

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Abstract

The objective of this work is the study of the asymptotic behavior of the solution, when $t \to +\infty$, of a class of parabolic equations. We show that if the initial condition is not null, the solution is exactly exponential when $t \to +\infty$ and the decrease rate is characterized by an element of the operator spectrum.

1 Introduction

It is well known that the solutions of the nonlinear equations of the type

$$\begin{cases}
  u_t + Au + f(u) = 0, \\
  u(0) = u_0,
\end{cases}$$

($A$ is an unbounded operator of the domain $D(A)$, and $f$ is a nonlinear operator) in $\Omega \times \mathbb{R}$ ($\Omega$ open bounded) associated with conditions at the edges are usually regular enough and convergent towards their state of equilibrium when $t \to +\infty$.

The objective of this work is to show that this convergence towards $u \equiv 0$ is exactly of the exponential type and the rate of this decrease is characterized by an eigenvalue of the operator $A$.

More precisely, we study the limit of the quotient $\frac{\int |A^{1/2}u|^2 \, dx}{\|u\|^2}$, denoted $\|u\|_{[u(x,t)]}^2$, and we show that it is an eigenvalue of the operator $A$, we deduce that there exists an eigensubspace such that $\frac{u(x,t)}{|u(x,t)|}$ is found concentrated under this subspace, and the solution $u(t)$ behaves exactly like the function $t \mapsto e^{-\Lambda t}$ when $t \to +\infty$ ($\Lambda \in \sigma(A)$).

2 Notations and recalls of certain results

Let $V$ and $H$ be two separable Hilbert spaces such that:

$$V \hookrightarrow H \text{ with compact injection,} \quad (2.1)$$
\[ V \text{ is dense in } H. \quad (2.2) \]

We denote by \( \| \cdot \| \) and \( | \cdot | \) the corresponding norms.

Consider the unbounded operator \( A \) with a range in \( H \):
\[ D(A) = \{ u \in V, \; Au \in H \}. \quad (2.3) \]

Supplying \( D(A) \) with the graph norm, \( A \) is then an isomorphism of \( D(A) \) in \( H \), so there exists a sequence of eigenvalues of \( A \)
\[ 0 < \lambda_1 < \lambda_2 < \cdots, \quad (2.4) \]
each with a finite multiplicity.

On the other hand, if \( R_j \) denotes the orthogonal projection onto the associated eigenspaces at \( j \), then
\[ R_jR_k = 0 \text{ if } i \neq j, \; R_1 \oplus R_2 \oplus \cdots = I. \quad (2.5) \]

We denote by
\[ 0 < \Lambda_1 < \Lambda_2 < \cdots < \Lambda_j < \cdots \quad (2.6) \]
the sequence of eigenvalues of multiplicity \( m_k \), and by \( \{S(t)\}_{t \geq 0} \) the nonlinear semi-group defined by
\[ S(t) : V \rightarrow V, \; u_0 \mapsto S(t)u_0. \quad (2.7) \]

Consider the problem given by
\[
(P) \begin{cases} 
  u_t + Au + f(u) = 0, \\
  u(0) = u_0,
\end{cases} \quad (2.8)
\]

\( A \) is an unbounded positive self-adjoint operator of the domain \( D(A) \), \( f(u) \) is a nonlinear operator.

From [3], we have the global existence results of uniform estimations in time as well as the asymptotic behavior when \( t \to +\infty \).

Indeed, if \( u_0 \in V \) and if \( f \) satisfies
\begin{enumerate}
  \item \( f \) continuous,
  \item \( (f(u) - f(v), u - v) + \lambda |u - v|^2 \geq 0 \; \forall u, v \in V, \lambda \in \mathbb{R} \),
  \item \( \exists \theta \in [0, 2[ \), \( \forall B \text{ bounded in } H, \)
\end{enumerate}
\[ |f(u)| \leq C_B \|u\|^{2-\theta} |Au|^{\theta/2} \; \forall u, v \in D(A), \quad (2.9) \]
the problem given by the system of equations (2.8) possesses a unique solution \( u \) that satisfies
\[ u \in C_b(\mathbb{R}_+, V) \cap L^2(0, \infty; D(A)), \quad (2.10) \]
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\[ u_t \in L^2(0, \infty; H) \]  
\[ (C_b = C^0 \cap L^{\infty}). \]

On the other hand, when $t \to +\infty$, the solution $u$ tends to its equilibrium state exponentially, and we have:

there exists a constant $c_0: \|u\| \leq C_0 e^{-\lambda(t-t_0)} \ \forall t \geq t_0, t_0 \geq 0.$  
\[ (2.12) \]

3 The behavior of the quotient $\frac{\|S(t)u_0\|^2}{|S(t)u_0|^2}$ when $t \to +\infty$

For $u_0 \in V, u_0 \neq 0$, the quotient

\[ \lambda(t) = \frac{\|S(t)u_0\|^2}{|S(t)u_0|^2} \]  
\[ (3.1) \]

is defined for $t \geq 0$. The behavior of $\lambda(t)$ is given by the following theorem.

**Theorem 3.1**

\[ \lim_{t \to +\infty} \lambda(t) = \Lambda(u_0), \]  
\[ (3.2) \]

where $\Lambda(u_0)$ is the eigenvalue of the operator $A$.

**Proof.** If $\lambda(t)$ is differentiated with respect to time, writing

\[ \frac{1}{2} \frac{d}{dt}(\lambda(t)) = \left(Au, \frac{du}{dt}\right) \frac{|u|^2}{|u|^4} - \left(\frac{du}{dt}, \lambda u\right) \frac{|u|^2}{|u|^4}, \]  
\[ (3.3) \]

i.e.,

\[ \frac{1}{2} \frac{d}{dt}(\lambda(t)) = \frac{1}{|u|^2} \left(Au - \lambda u, \frac{du}{dt}\right), \]  
\[ (3.4) \]

or $\frac{du}{dt} = -(Au - f(u))$, and (3.4) will be

\[ \frac{1}{2} \frac{d}{dt}(\lambda(t)) = \frac{1}{|u|^2} (Au - \lambda u, Au + f(u)). \]  
\[ (3.5) \]

However, we have $(Au - \lambda u, \lambda u) = (Au - \frac{\|u\|^2}{|u|^2} \cdot u, \frac{\|u\|^2}{|u|^2} \cdot u) = 0$.  
So we can write using (3.5):

\[ \frac{1}{2} \frac{d}{dt}(\lambda(t)) = -\frac{1}{|u|^2} (Au - \lambda u, Au - \lambda u) + (Au - \lambda u, f(u)). \]  
\[ (3.6) \]
Then by putting \( v(t) = \frac{u(t)}{|u(t)|} \),

\[
\frac{1}{2} \frac{d}{dt}(\lambda(t)) + |(A - \lambda)v|^2 = -((A - \lambda)v, \frac{f(u)}{|u|}).
\]

(3.7)

Majorizing the term on the right-hand side and applying the inequality of Young, we obtain

\[
\frac{d}{dt}(\lambda(t)) + |(A - \lambda)v|^2 \leq \frac{|f(u)|^2}{|u|^2}.
\]

(3.8)

Taking into account \(|f(u)| \leq \eta(u) \|u\|\) with \(\eta(u) = \mathcal{O}(e^{-\alpha t})\), \(\alpha > 0\), we have:

\[
\frac{d}{dt}(\lambda(t)) + |(A - \lambda)v|^2 \leq c_1 e^{-\alpha t} \lambda(t), \quad \alpha > 0.
\]

(3.9)

Omitting the term \(|(A - \lambda)v|^2\) in (3.9), we get

\[
\frac{d}{dt}(\lambda(t)) \leq c_1 e^{-\alpha t} \lambda(t),
\]

(3.10)

which will be integrated:

\[
\Lambda_1 \leq \lambda(t) \leq \lambda(t_0) e^{\int_{t_0}^{t} e^{-\alpha s} ds}, \quad t > t_0.
\]

(3.11)

Taking the upper limit as \(t \to +\infty\), it will be

\[
\Lambda_1 \leq \limsup_{t \to +\infty} \lambda(t) = \lambda(t_0) e^{\int_{t_0}^{+\infty} e^{-\alpha s} ds} < +\infty,
\]

(3.12)

then the lower limit as \(t \to +\infty\),

\[
\Lambda_1 \leq \liminf_{t \to +\infty} \lambda(t) \leq \liminf_{t \to +\infty} \lambda(t) < +\infty.
\]

(3.13)

We deduce then that \(\lambda(t)\) converges toward a limit \(\Lambda(u_0)\). Moreover, we have:

\[
\lambda(t) \geq \Lambda(u_0) e^{\int_{t_0}^{+\infty} e^{-\alpha s} ds}.
\]

(3.14)

To show that \(\Lambda(u_0) \in \sigma(A) = \{\Lambda_1, \Lambda_2, \ldots\}\), we take again and integrate the inequality (3.9), it becomes then

\[
\lambda(t) - \lambda(t_0) + \int_{t_0}^{t} |(A - \lambda)v|^2(s) \, ds \leq \sup_{t \geq t_0} \lambda(t) \int_{t_0}^{t} c_1 e^{-\alpha s} \, ds, \quad \alpha > 0.
\]

(3.15)
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It results from it that

\[ |(A - \lambda)v| \in L^2(t_0, \infty). \]  

(3.16)

We then deduce that there exists a sequence \( t_j \to +\infty \) such that

\[ |(A - \lambda)(t_j)v(t_j)| \to 0 \quad \text{when} \quad t_j \to +\infty. \]  

(3.17)

Thus,

\[ |(A - \Lambda(u_0))v(t_j)| \to 0 \quad \text{when} \quad t_j \to +\infty. \]

The sequence \(|(Av(t_j))|\) is bounded, by an \( A^{-1} \) capacity, we are sure of the existence of a subsequence, denoted by \( (t_j) \), such that

\[ v(t_j) \to \overline{v} \quad \text{in} \quad H \quad \text{strongly and in} \quad V \quad \text{weakly}. \]  

(3.18)

It results from it that

\[ Av(t_j) = (A - \lambda(t_j))(v(t_j)) + (\lambda(t_j) - \Lambda(u_0))(v(t_j)) + \Lambda(u_0)(v(t_j)) \to \Lambda(u_0)v, \]  

(3.19)

with \( \overline{v} \in D(A) \) (A is maximum accretive, so closed).

Moreover, \(|v(t_j)| = 1\). Thus \(|\overline{v}| = 1\), it results from (3.17) and (3.20):

\[ Av = \Lambda v \]  

(3.20)

(Thus \( \lambda \) is an eigenvalue of \( A \)).

**Corollary 3.1**

\[ \lim_{t \to +\infty} \frac{\log \|u(t)\|}{t} = \lim_{t \to +\infty} \frac{\log u(t)}{t} = -\Lambda(u_0). \]  

(3.21)

Let us use the equation

\[ \frac{1}{2} \frac{d}{dt} |u|^2 + \|u\|^2 + (f(u), u) = 0. \]  

(3.22)

According to Theorem 3.1, we deduce that there exists \( t_\varepsilon \ (\varepsilon > 0) \) such that \( \forall t > t_\varepsilon \) we have

\[ -\varepsilon + \Lambda(u_0) \leq \frac{\|u(t)\|^2}{|u(t)|^2} \leq \Lambda(u_0) + \varepsilon. \]  

(3.23)

From (3.23) and (3.22), taking into account \(|f(u)| \leq \eta(t) \|u\|, \eta(t) = \circ(e^{-\alpha t}), \alpha > 0, \) it follows that

\[ \{-c_1 e^{-\alpha t} - 2(\varepsilon + \Lambda(u_0))\} |u|^2 \leq \frac{d}{dt} |u|^2 \leq \{c_1 e^{-\alpha t} - 2(\Lambda(u_0) + \varepsilon)\} |u|^2, \]  

(3.24)
and from it, that

\[-(\varepsilon_1 + \Lambda(u_0)) \leq \frac{\log |u(t)|}{t} \leq (\varepsilon_1 - \Lambda(u_0)), \quad \forall t > t_\varepsilon, (3.25)\]

where

\[\lim_{t \to +\infty} \frac{\log |u(t)|}{t} = -\Lambda(u_0), \quad (3.26)\]

and since \(\frac{\|u(t)\|}{|u(t)|}\) is bounded, thus we have

\[\lim_{t \to +\infty} \frac{\log \|u(t)\|}{t} = -\Lambda(u_0). \quad (3.27)\]

4 The behavior of \(S(t)u_0\) when \(t \to +\infty\)

We recall that for the eigenvalue \(\lambda\) of the operator \(A\), we denote by \(R_\lambda\) the orthogonal projection onto the eigenspace associated with \(\lambda\):

\[R_\lambda \omega = \sum_{\lambda_j=\lambda} (\omega, \omega_j) \omega_j. \quad (4.1)\]

As well as if \(\lambda_m < \lambda < \lambda_{m+1}\) and \(\Lambda_M < \lambda < \Lambda_{M+1}\), then

\[R_\lambda = P_M - P_m \quad (4.2)\]

(Where \(P_m\) is an orthogonal projection).

The following propositions show that the solution \(u\) behaves asymptotically like \(R_\Lambda u\).

**Proposition 4.1**

\[
\lim_{t \to \infty} \frac{\|I - R_\Lambda(u_0)\|}{\|S(t)u_0\|} = \lim_{t \to \infty} \frac{\|I - R_\Lambda(u_0)\|}{\|S(t)u_0\|} = 0. \quad (4.3)
\]

We start by establishing a lemma that specifies Corollary 3.1.

**Lemma 4.1** There exist two positive constants \(c_1\) and \(c_2\) such that

\[|u(t)| = c_1 e^{-\Lambda(u_0)(t-t_0)} \quad \forall t \geq t_0, \quad (4.4)\]

\[\|u(t)\| = c_2 e^{-\Lambda(u_0)(t-t_0)} \quad \forall t \geq t_0. \quad (4.5)\]
Proof. Since \( \|u(t)\| \) is bounded, it is sufficient to prove (4.4). Suppose the inequality

\[
\frac{1}{2} \frac{d}{dt} |u|^2 + \|u\|^2 \leq \eta(t) \lambda^{1/2} |u|^2.
\]

(4.6)

By virtue of \( \|u\|^2 \geq \lambda(t) |u|^2 \) and (4.6) we obtain

\[
\frac{d}{dt} \log(\|u\|^2 e^{2\Lambda t}) \leq 2 \eta \lambda^{1/2} + 2 (\lambda - \Lambda).
\]

(4.7)

From (3.2) and (4.7) there follows (4.4).

Proof of Proposition 4.1. We start by establishing an evolution equation for \( v(t) = \frac{S(t)}{\|S(t)\|} = \frac{u(t)}{|u(t)|} \),

\[
\frac{dv}{dt} = \frac{1}{|u(t)|} \frac{du}{dt} - \frac{1}{|u(t)|^2} \frac{d}{dt} |u(t)| \cdot u,
\]

(4.8)

but \( 2 \left( \frac{du}{dt}, u \right) = 2 |u| \frac{d}{dt} |u| = \frac{d}{dt} |u|^2 \); \( \frac{du}{dt} = -(Au + f(u)) \) and (4.8) takes the form

\[
\frac{dv}{dt} + (A - \lambda)v = \frac{1}{|u(t)|^2} (f(u), u) \cdot v - \frac{f(u)}{|u(t)|}.
\]

(4.9)

We put

\[
\rho = \frac{1}{|u(t)|^2} \left| f(u), u \right| \cdot v - \frac{f(u)}{|u(t)|}.
\]

(4.10)

We deduce from the estimation of \( f(u) \) and of the bounded \( \lambda(t) \):

\[
\rho \leq \frac{|f(u)|}{|u|^2} \|u\| \|v\| + \frac{f(u)}{|u|} \leq \sup \lambda(t)c_1 e^{-\alpha t}, \quad \alpha > 0,
\]

(4.11)

i.e.,

\[
\rho \leq c_5 e^{-\alpha t}, \quad \alpha > 0.
\]

We denote \( q = (I - P\lambda)v, \) thus if we apply \( (I - P\lambda) \) to (4.9) and we take the scalar product in \( H \) of the results by \( Aq \), we have, using (4.10),

\[
\frac{1}{2} \frac{d}{dt} \|q\|^2 + |Aq|^2 - \lambda(t) \|q\|^2 \leq \rho |Aq|.
\]

(4.12)

Applying the inequality \( 2\rho \|Aq\| \leq 2\varepsilon \|Aq\|^2 + \frac{\varepsilon^2}{2}e^\varepsilon (\varepsilon > 0) \), we obtain

\[
\frac{1}{2} \frac{d}{dt} \|q\|^2 + 2(1 - \varepsilon) \|Aq\|^2 - 2\lambda(t) \|q\|^2 \leq c_6 e^{-\alpha t}, \quad \alpha > 0.
\]

(4.13)
Denoting by $\Lambda' > \Lambda$ the first eigenvalue that is strictly greater than $\Lambda$, $\delta = \Lambda' - \Lambda > 0$, and since $|Aq|^2 \geq \Lambda' \|q\|^2$, we deduce from (4.13)

$$\frac{d}{dt} \|q\|^2 + 2 [ (1 - \varepsilon)\Lambda - \Lambda ] \|q\|^2 \leq c_6 e^{-\alpha t}, \quad \alpha > 0. \tag{4.14}$$

Choosing $\varepsilon = \frac{\delta}{2\Lambda'}$, (4.14) will become

$$\frac{d}{dt} \|q\|^2 + \delta \|q\|^2 \leq c_6 e^{-\alpha t}, \quad \alpha > 0, \tag{4.15}$$

that will be integrated:

$$\|q(t)\|^2 \leq \|q(t_0)\|^2 e^{-\delta(t-t_0)} + e^{-\delta} \int_{t_0}^{t} e^{(\alpha-\delta)s} ds. \tag{4.16}$$

We deduce then

$$\|q(t)\| = \|(I - P_{\Lambda'})v\| \to 0 \quad \text{as} \quad t \to +\infty, \tag{4.17}$$

$$\int_0^{+\infty} \|q(t)\|^2 \, dt < +\infty. \tag{4.18}$$

On the other hand, denoting $\Psi(t) = P_{\Lambda''}w$, where $\Lambda'' < \Lambda$, $\delta'' = \Lambda - \Lambda'' > 0$, the scalar product of $P_{\Lambda''}$ applied to (4.9) by $\Psi(t)$ will be written as

$$\frac{1}{2} \frac{d}{dt} |\Psi|^2 + \|\Psi\|^2 - \lambda(t) |\Psi|^2 \geq -\rho |\Psi|^2. \tag{4.19}$$

Applying the inequality of Young to the right-hand side, (4.19) becomes, using also $\|\Psi\|^2 \geq \Lambda'' |\Psi|^2$:

$$\frac{d}{dt} |\Psi|^2 \geq (2\lambda(t) - 2(1 + \varepsilon')\Lambda'') |\Psi|^2 - c_7 e^{-\alpha t}, \tag{4.20}$$

or, according to $\lambda(t) \to \Lambda(u_0)$ as $t \to +\infty$, we can write for $t \gg 1$

$$\frac{d}{dt} |\Psi|^2 \geq 2(\Lambda - (1 + \varepsilon')\Lambda'') |\Psi|^2 - c_7 e^{-\alpha t}. \tag{4.21}$$

Choosing $\varepsilon' = \frac{\delta'}{2\Lambda''}$, (4.21) will be written as

$$\frac{d}{dt} |\Psi|^2 \geq \delta' |\Psi|^2 - c_7 e^{-\alpha t}, \tag{4.22}$$
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that will be integrated:

\[
|\Psi|^2 + \frac{e^{\delta t}}{(\delta + \alpha)}(e^{-(\alpha-\delta)t_0} - e^{-(\alpha-\delta)t}) \geq |\Psi(t_0)|^2 e^{\delta(t-t_0)}.
\]

(4.23)

Then, multiplying (4.23) by \( e^{-\delta(t-t_0)} \), for \( t \to +\infty \) we obtain

\[
|\Psi(t_0)| \leq c_8 e^{-\delta t_0}, \quad t_0 \geq 0.
\]

(4.24)

We deduce from (4.24):

\[
|\Psi(t)| = |P_\Lambda^0 v(t)| \to 0 \quad \text{as} \quad t \to +\infty,
\]

(4.25)

\[
\int_0^{+\infty} |P_\Lambda^0 v|^2(s) \, ds < +\infty.
\]

(4.26)

Since \((I - R_\Lambda)v = P_\Lambda^0 v + (I - P_\Lambda)v\), from (4.17) and (4.25) there results (4.3).

Corollary 4.1

\[
\int_0^{+\infty} \|\Lambda^0 v\|^2(s) \, ds < +\infty.
\]

(4.27)

Proof. (4.27) results from (4.18) and (4.26).

The following corollary makes precise the convergence of \( \|v(t)\|^2 \) towards \( \Lambda(u_0) \).

Corollary 4.2

\[
\int_0^{+\infty} |\lambda(t) - \Lambda(u_0)| \, dt < +\infty.
\]

(4.28)

Proof. We note that \( |\lambda - \Lambda(u_0)| \leq (1 + \frac{1}{\Lambda^2}) \|\Lambda - (I - R_\Lambda)v\|^2 \), thus from (4.27) we deduce (4.28).

Theorem 4.1 \( \lim_{t \to -\infty} e^{\Lambda(u_0) t} |S(t)u_0| \) exists, is finite and not null.

More precisely: \( e^{\Lambda(u_0) t} S(t)u_0 \) converges in \( H \) and \( V \) towards the eigenvector \( \Lambda(u_0) \) of \( A \) associated with \( \Lambda(u_0) \).

Proof of Theorem 4.1. Applying \( R_\Lambda \) to the equation \( \frac{du}{dt} + Au + f(u) = 0 \), it becomes

\[
\frac{d}{dt}(e^{\Lambda t} R_\Lambda u(t)) e^{\Lambda t} R_\Lambda f(u) = 0,
\]

(4.29)
that will be integrated:

\[ e^{\Lambda t} R_\Lambda u(t) - e^{\Lambda s} R_\Lambda u(s) = - \int_s^t e^{\Lambda \sigma} R_\Lambda f(u(\sigma)) \, d\sigma, \]  

(4.30)

but \(|f(u)| \leq c_9 e^{-(\Lambda+\alpha)t}\) according to (4.5) and then

\[ \left| \int_s^t e^{\Lambda \sigma} R_\Lambda f(u(\sigma)) \, d\sigma \right| \leq c_1 \int_s^t e^{-\alpha \sigma} \, d\sigma \leq c_1 \int_s^t e^{-\alpha \sigma} \, d\sigma < +\infty. \]  

(4.31)

Consequently, the integral on the right-hand side of (4.30) is convergent and \(e^{\Lambda t} R_\Lambda u(t)\) converges.

If \(\lim_{t \to +\infty} e^{\Lambda t} R_\Lambda u(t) = U_\Lambda = 0\), then the equation (4.30) will be:

\[ e^{\Lambda t} R_\Lambda u(t) = \int_t^{+\infty} e^{\Lambda \sigma} R_\Lambda f(u(\sigma)) \, d\sigma, \]  

(4.32)

and since \(|f(u)| \leq c_9 e^{-(\Lambda+\alpha)t}\), \(\alpha > 0\), we deduce that

\[ |R_\Lambda u(t)| \leq c_{10} e^{-(\Lambda+\alpha)t}, \]  

(4.33)

but (4.33) contradicts Corollary 3.1, since according to Proposition 4.1 we have \(|R_\Lambda u(t)|\) is equivalent to \(c|u(t)|\) when \(t \to \infty\); consequently, the limit \(U_\Lambda(u_0) \neq 0\), and we have

\[ \lim_{t \to +\infty} e^{\Lambda t} R_\Lambda u(t) = U_\Lambda \neq 0. \]  

(4.34)

We simply verify that \(U_\Lambda\) is an eigenvalue of the operator \(A\).

Because of Proposition 4.1, we deduce that \(R_\Lambda u(t) = u(t) + \varepsilon(t) u(t)\), where \(\varepsilon(t) \to 0\) \((t \to +\infty)\) and we obtain

\[ \lim_{t \to +\infty} e^{\Lambda t} u(t) = U_\Lambda \neq 0, \]  

(4.35)

which leads to the proof of Theorem 4.1.

**Remark 4.1** Theorem 4.1 proves that for \(t \to +\infty\), \(|u(t)| = O(e^{-\mu t}) \ \forall \mu > 0\), so \(u \equiv 0\).
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References


